Harmonic Algebraic Curves and Noncrossing Partitions

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Overview

1. Gauss's proof of the Fundamental Theorem of Algebra

2. Noncrossing matchings, bimatchings, and basketballs

3. The Inverse Basketball Theorem

4. Necklaces of basketballs

1. Gauss's Proof of the FTA

Fundamental Theorem of Algebra:
Every complex polynomial $f(z)$ of degree n has exactly n complex roots (counting multiplicities).

Proof. (Gauss 1799; Gersten–Stallings 1988)

Consider the plane algebraic curves

$$\boldsymbol{R} = \{z : \ \operatorname{Re} f(z) = 0\},$$
$$\boldsymbol{I} = \{z : \ \operatorname{Im} f(z) = 0\},$$

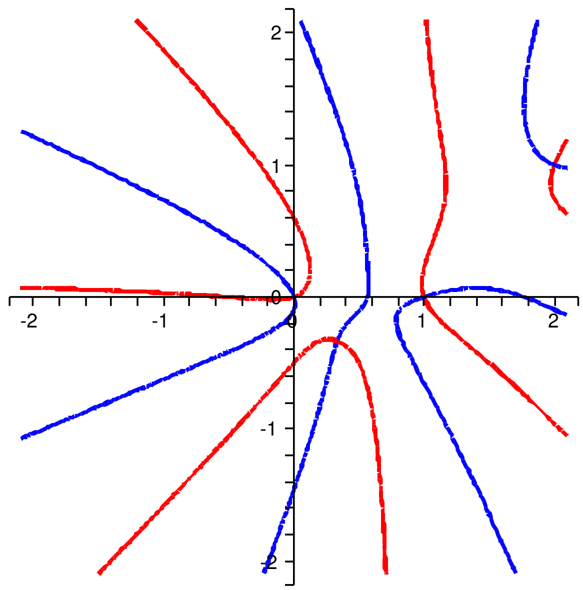
$\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ are polynomials in x and y, and

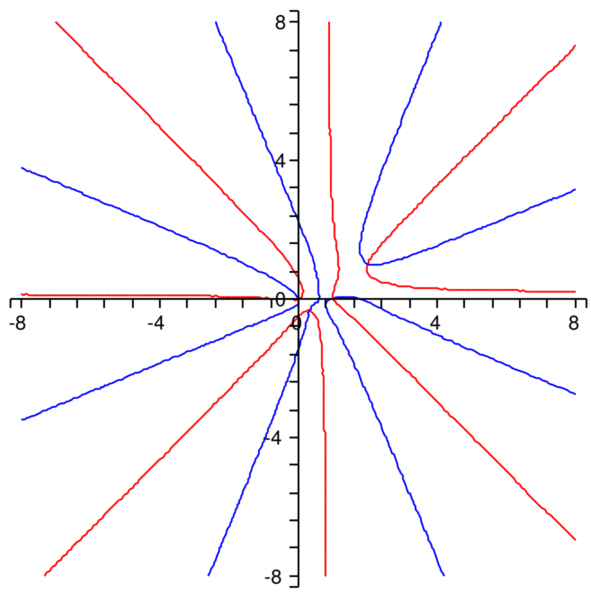
$$\boldsymbol{R} \cap \boldsymbol{I} = \{z : \ f(z) = 0\}.$$

In polar coordinates $z = re^{i\theta}$,

$$\operatorname{Re} f(z) = r^n \cos n\theta + O(r^{n-1}),$$
$$\operatorname{Im} f(z) = r^n \sin n\theta + O(r^{n-1}).$$

Example: $f(z) = z(z-1)(z-(2+i))(z-(\frac{1-i}{3}))$





Choose $\rho \in \mathbb{R}$ greater than the magnitude of every root of f.

Let D be a disk of radius $\gg \rho$, and $S = \partial D$ its boundary circle.

$\boldsymbol{R} \cap S$ consists of $2n$ points whose arguments are approximately the zeroes of $\cos n\theta$, i.e.,

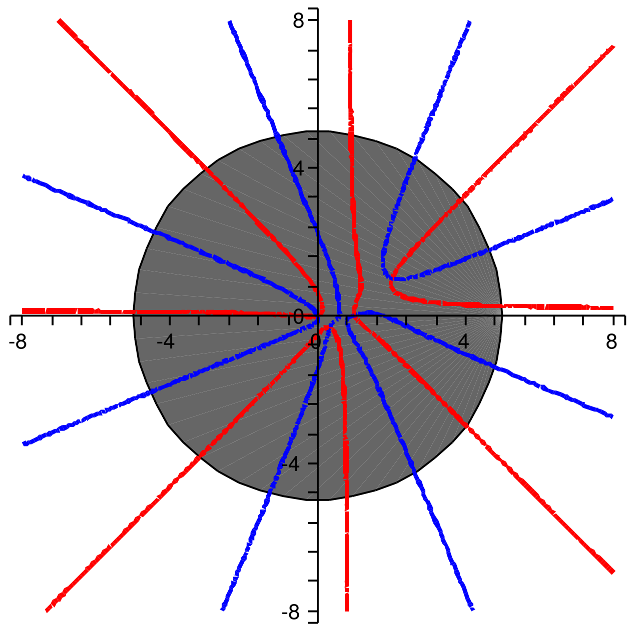
$$\boldsymbol{R} \cap \partial S \;\approx\; \left\{\rho,\; \rho e^{i\pi/n},\; \rho e^{2i\pi/n},\; \ldots,\; \rho e^{(2n-1)i\pi/n}.\right\}$$

Similarly,

$$\boldsymbol{I} \cap \partial S \;\approx\; \left\{\rho e^{i\pi/2n},\; \rho e^{3i\pi/2n},\; \ldots,\; \rho e^{(4n-1)i\pi/2n}.\right\}$$

Outside D, the curve $\boldsymbol{R}$ (resp. $\boldsymbol{I}$) consists of $2n$ disjoint half-branches, asymptotic to the lines $\theta = k\pi/2n$ with n even (resp. odd).

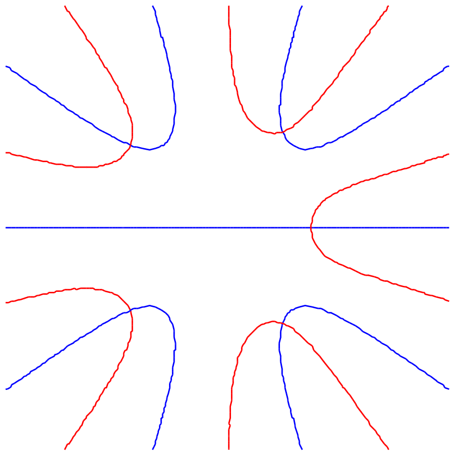
Each half-branch of $\boldsymbol{R}$ (resp. $\boldsymbol{I}$) must connect with another one inside D to make n full branches.



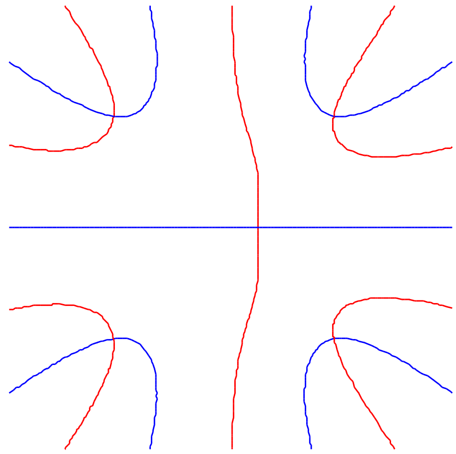
Each of the n branches of $\boldsymbol{R}$ has an odd number of half-branches of $\boldsymbol{I}$ on either side of it, hence must meet some branch of $\boldsymbol{I}$.

That is, $f(z)$ has at least n roots!

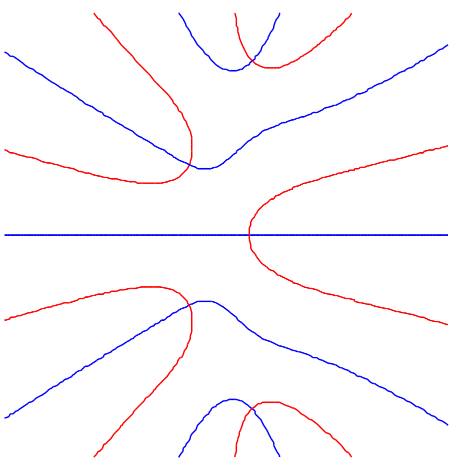
Possibilities for the curves *R* and *I*



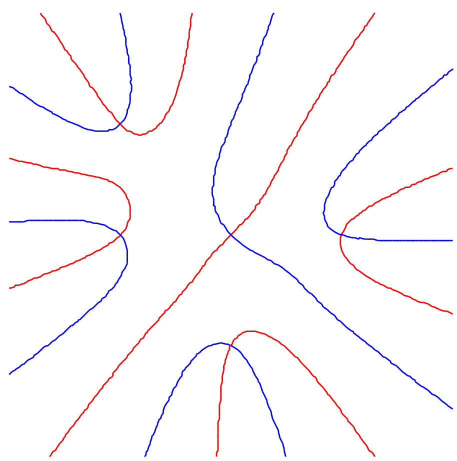
$z^5 + z + 1$



$z^5 + z^2 + 1$



$z^5 + 6z^3 + 3z^2 + 5z - 2$



$z(z-1)(z+1)(z+i)(z+1-i)$

2. Noncrossing Partitions and Matchings

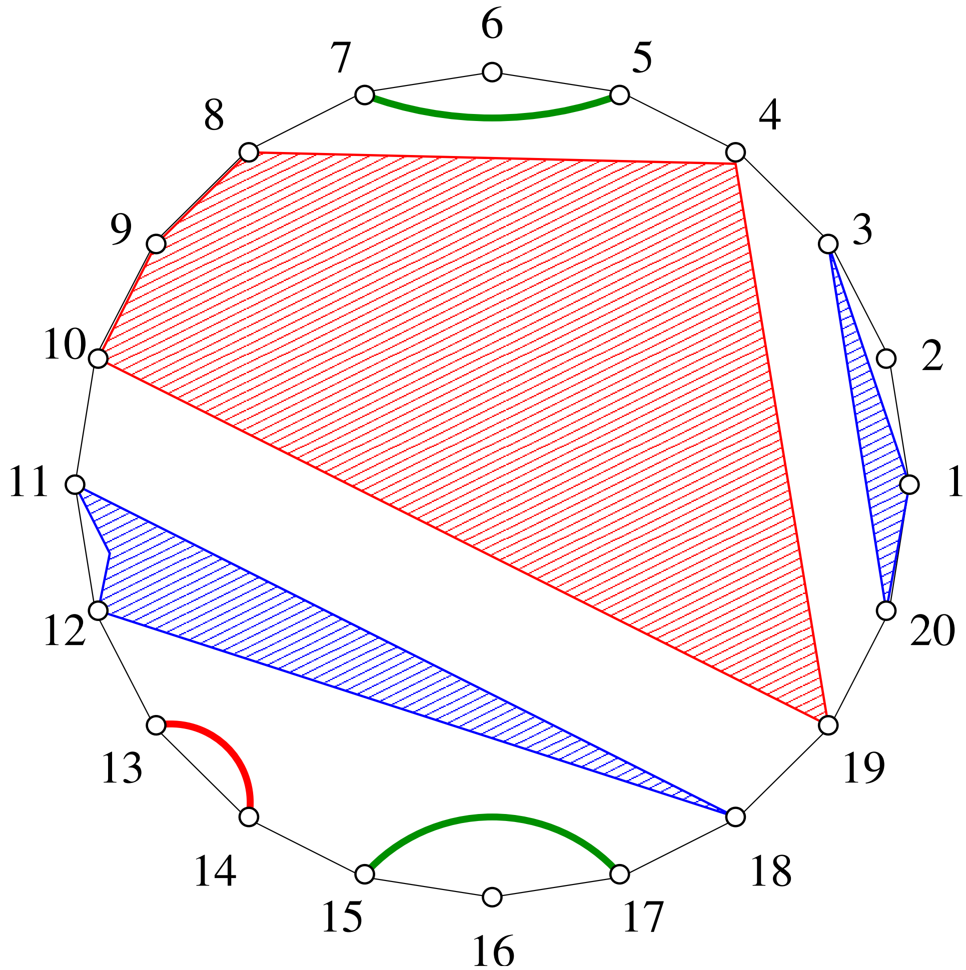
Let $V \subset \mathbb{N}$ be a finite set of vertices.

A **partition** of V is a collection of pairwise disjoint sets $V_1, \ldots, V_k$ ("blocks") with $\bigcup V_i = V$.

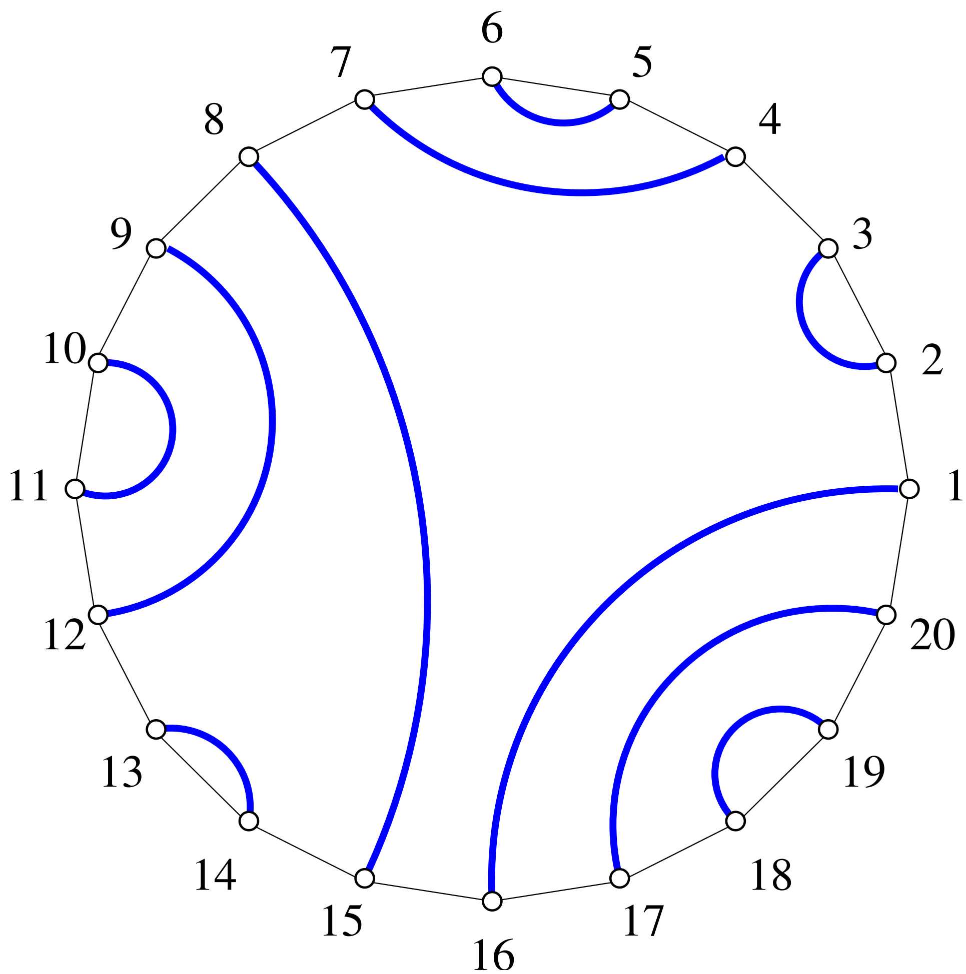
Two blocks V_a, V_b **cross** if for some $i < j < k < \ell \in S$,

$$i, k \in S_a \qquad \text{and} \qquad j, \ell \in S_b.$$

The partition is **noncrossing** if no two blocks cross.



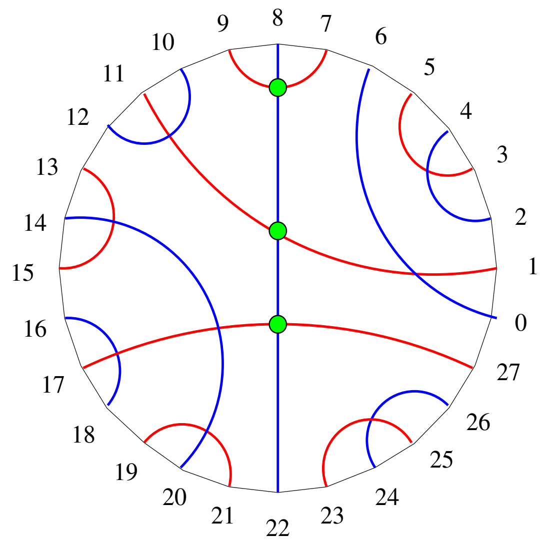
A **noncrossing matching (NCM) of order n** is a noncrossing partition of $[2n]$ in which every block has size 2.



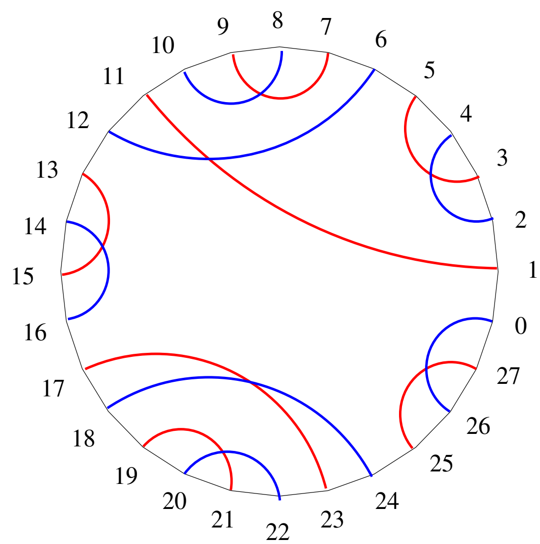
- Every NCM pairs even vertices with odd vertices.
- $\#\{\text{NCMs of order } n\} = \dfrac{1}{n+1}\dbinom{2n}{n}.$

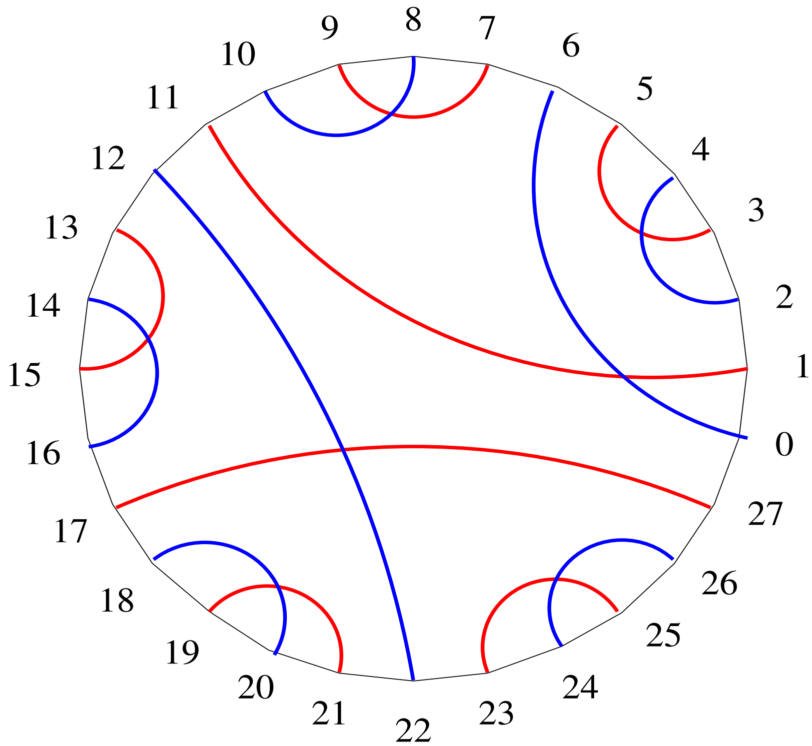
Bimatchings and Basketballs

Let $B^{\mathbf{e}}$, $B^{\mathbf{o}}$ be noncrossing matchings on $\{0, 2, 4, \ldots, 2n-2\}$ and $\{1, 3, 5, \ldots, 2n-1\}$ respectively. The pair $B = (B^{\mathbf{e}}, B^{\mathbf{o}})$ is called a (noncrossing) **bimatching** (of order n).

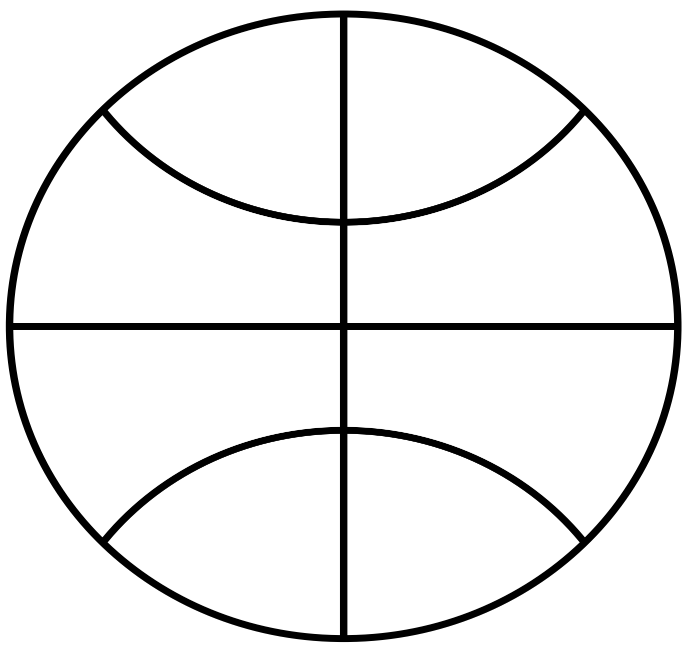


Every pair of $B^{\mathbf{e}}$ crosses an odd number of pairs of $B^{\mathbf{o}}$, and vice versa. B is called a **basketball** if every pair of $B^{\mathbf{e}}$ crosses *exactly one* pair of $B^{\mathbf{o}}$.





A combinatorial basketball

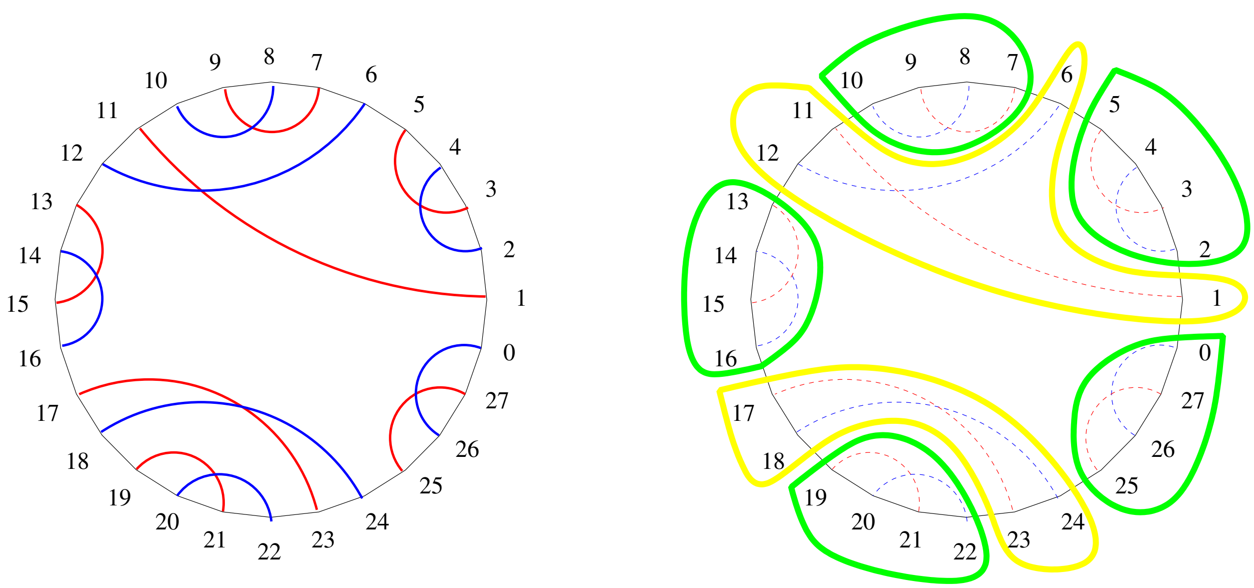


An NBA–approved basketball

Proposition The number of n-basketballs is

$$b(n) = \frac{1}{3n+1}\binom{4n}{n}.$$

Sketch of proof: $b(n)$ counts noncrossing partitions of $4n$ vertices into n 4-blocks [Edelman 1980]. There is a bijection between n-basketballs and such partitions.



Other objects enumerated by $b(n)$:

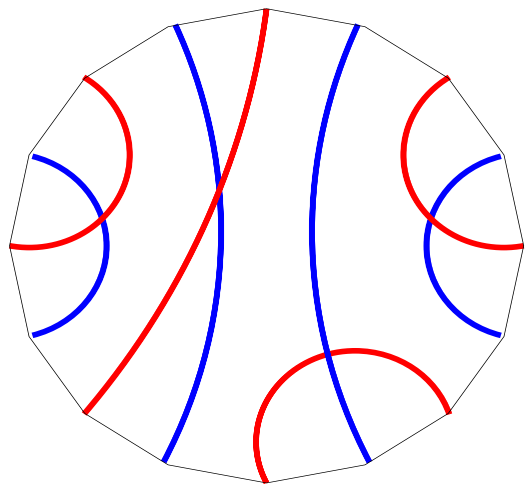
- plane quaternary trees with n internal vertices
- dissections of a $(3n+2)$-gon into n pentagons
- certain rooted plane maps [Liskovets-Walsh]

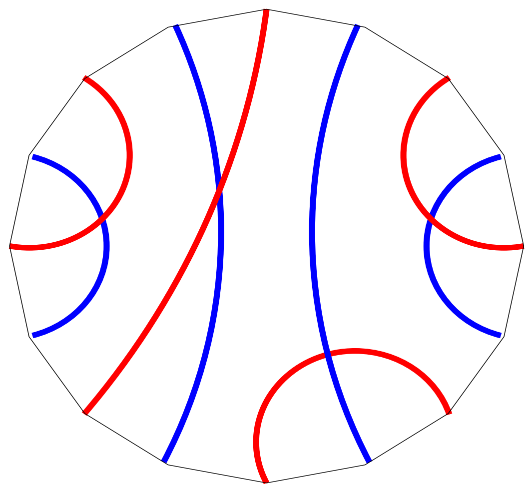
n	1	2	3	4	5	6	...
$b(n)$	1	4	22	140	969	7084	...

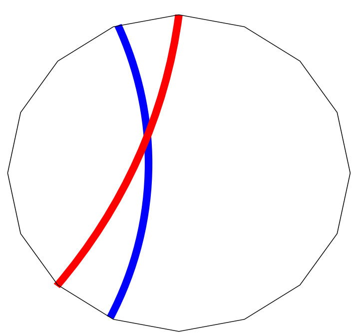
- For $n \geq 2$, every n-basketball contains at least two **ears**, or pairs of pairs

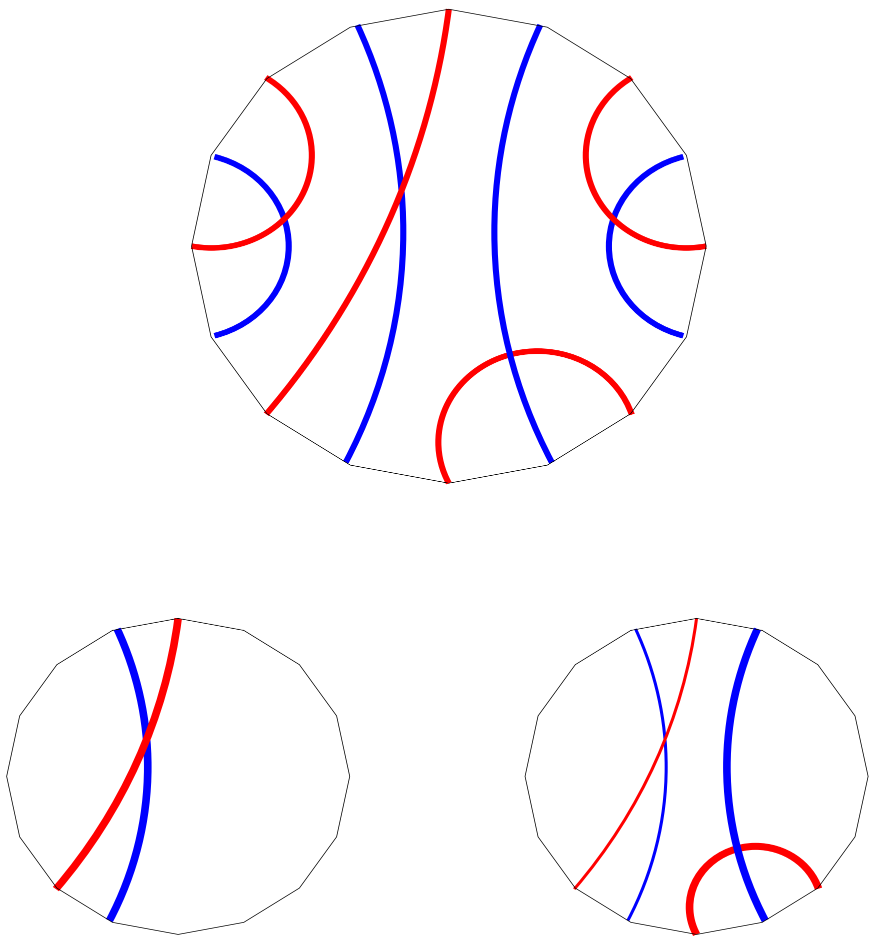
$$\{i, i+2\}, \{i+1, i+3\}.$$

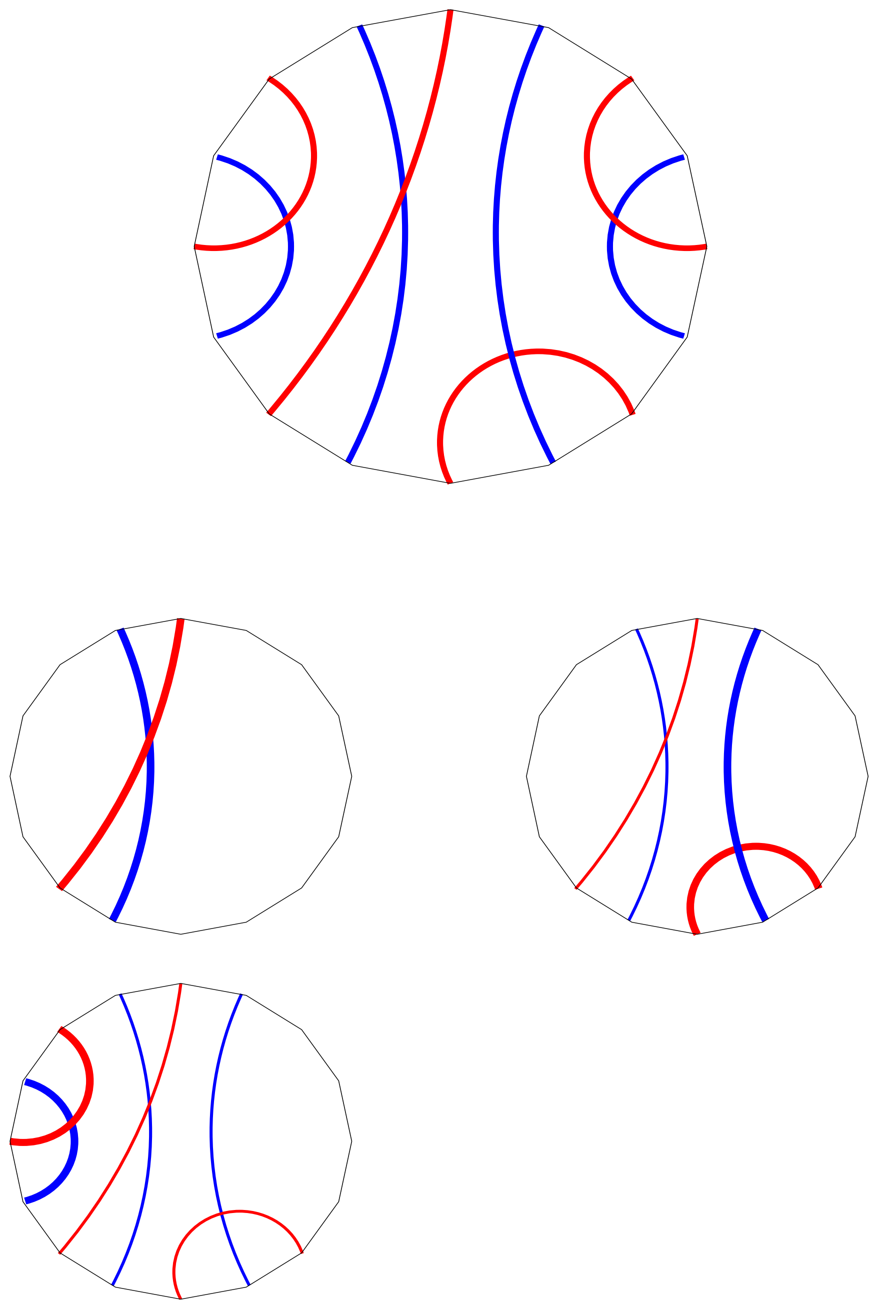
- Every basketball can be built up inductively by adding ears, one at a time.

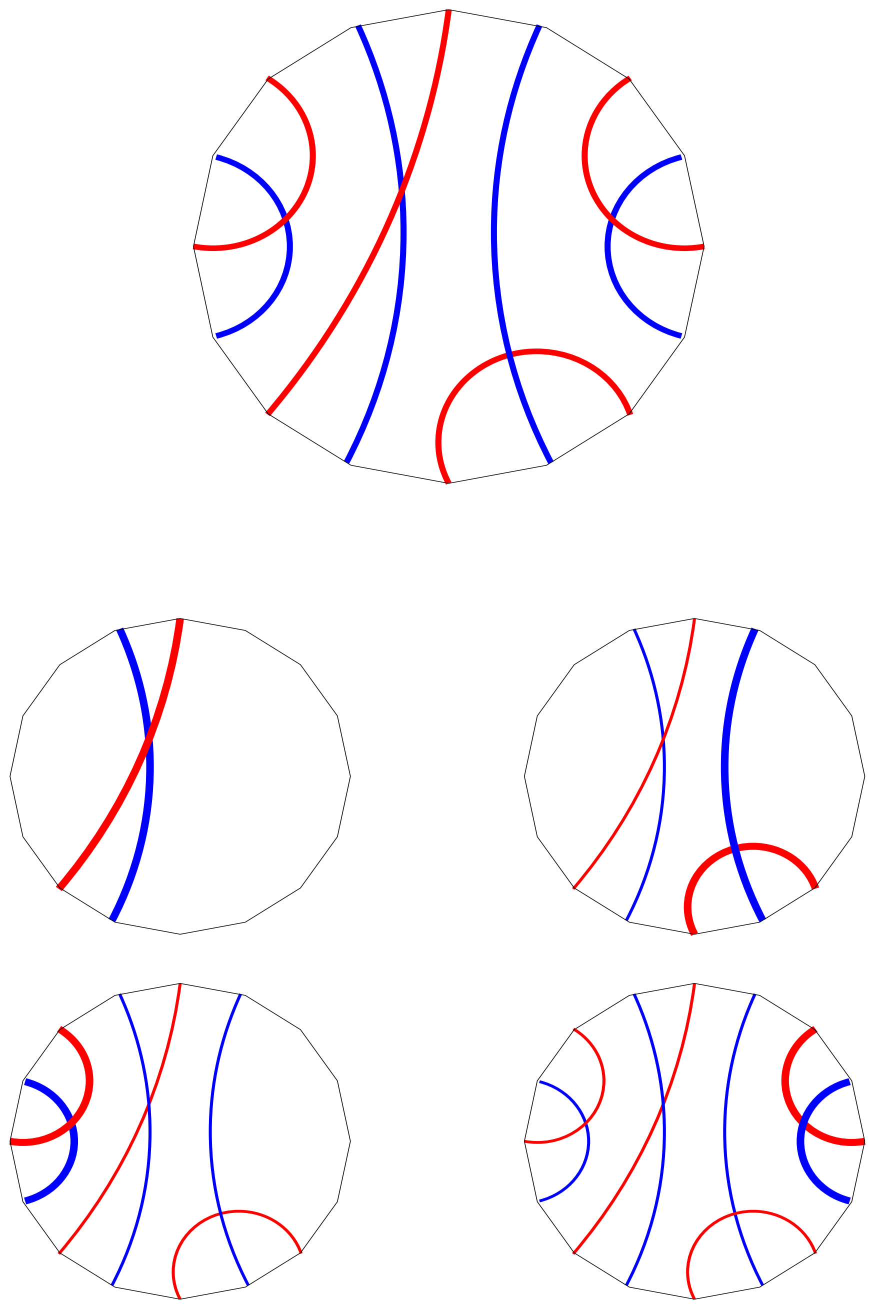












The basketball of a complex polynomial

Let $f(z)$ be a complex polynomial of degree n, and let $\theta \in \mathbb{R}$. Define

$$C_\theta(f) \;=\; \{z \;\mid\; \operatorname{Im}(e^{-i\theta} f(z)) = 0\}.$$

So $\boldsymbol{R} = C_{\pi/2}(f)$, $\boldsymbol{I} = C_0(f)$.

Fact $C_\theta(f)$ is nonsingular for all but $(n-1)$ values of θ.

When $C_\theta(f)$ is nonsingular, it gives rise to a well-defined noncrossing matching $M(f,\theta)$.

When $\boldsymbol{R}$, $\boldsymbol{I}$ are both nonsingular, they determine a basketball

$$B(f) \;=\; \big(M(\boldsymbol{R}), M(\boldsymbol{I})\big).$$

The Inverse Basketball Theorem

Let $0 \leq \alpha < \beta \leq \pi$,

and let $B = (B^{\mathbf{e}}, B^{\mathbf{o}})$ be any (combinatorial) n-basketball.

Then there exists a polynomial f of degree n such that

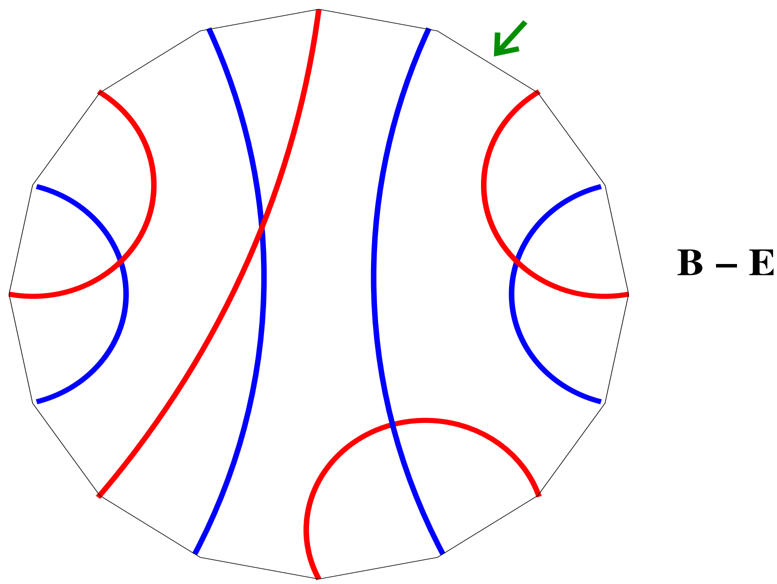
$$B^{\mathbf{e}} = M(f, \alpha) \qquad \text{and} \qquad B^{\mathbf{o}} = M(f, \beta).$$

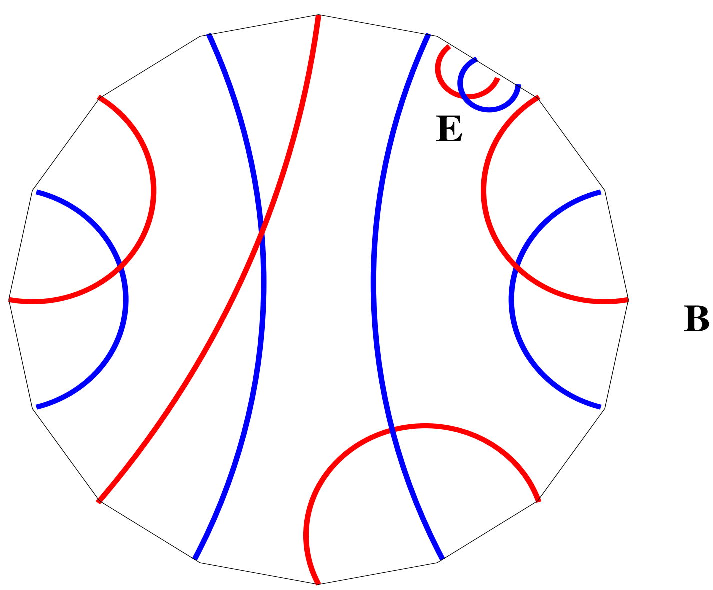
(In particular, there exists a polynomial f such that $B = B(f)$.)

Sketch of the proof

Given a basketball B of order n, choose an ear $E \subset B$.

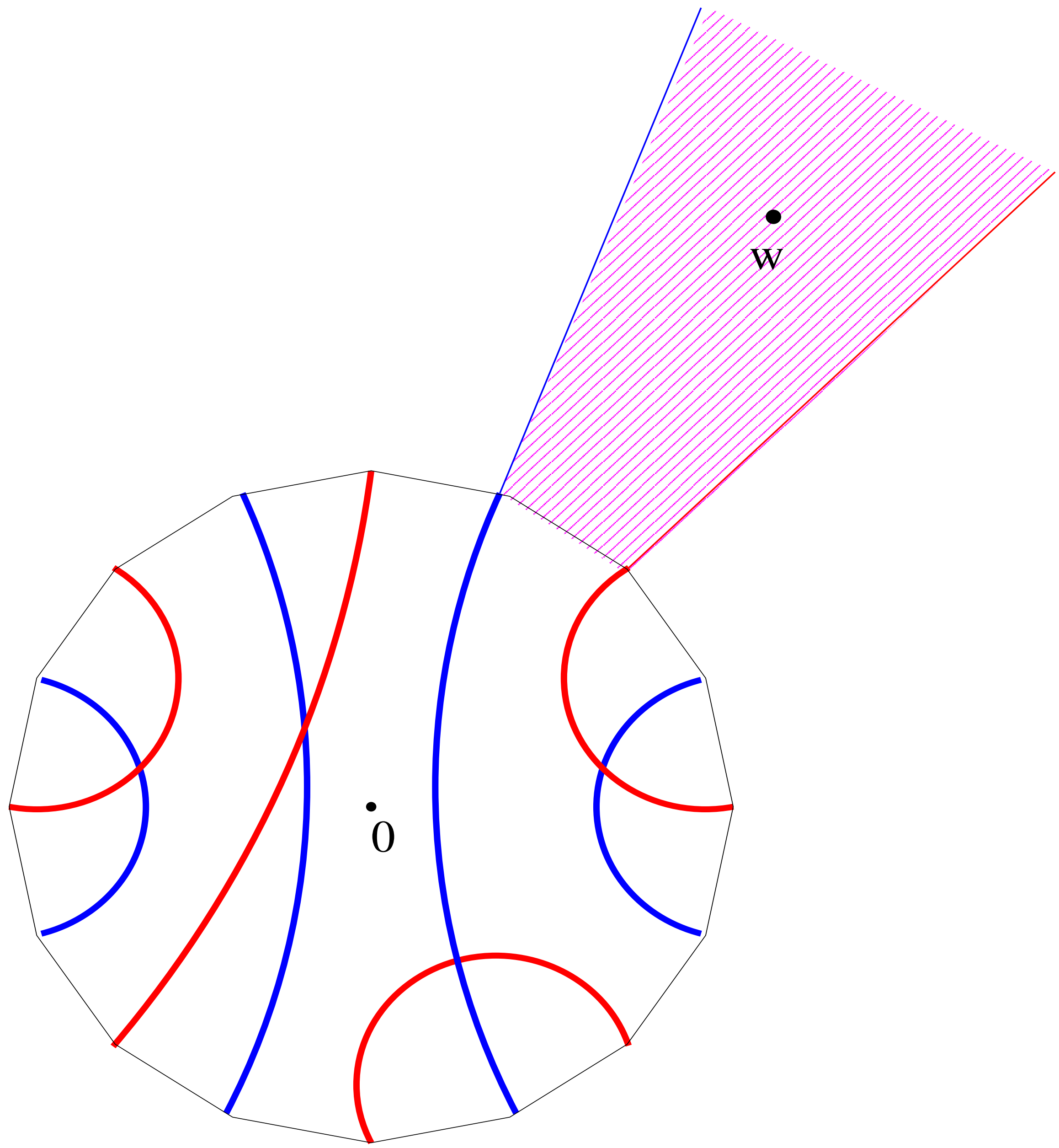
By induction on n, there is a polynomial $f(z)$ with $B(f) \;=\; B - E$.



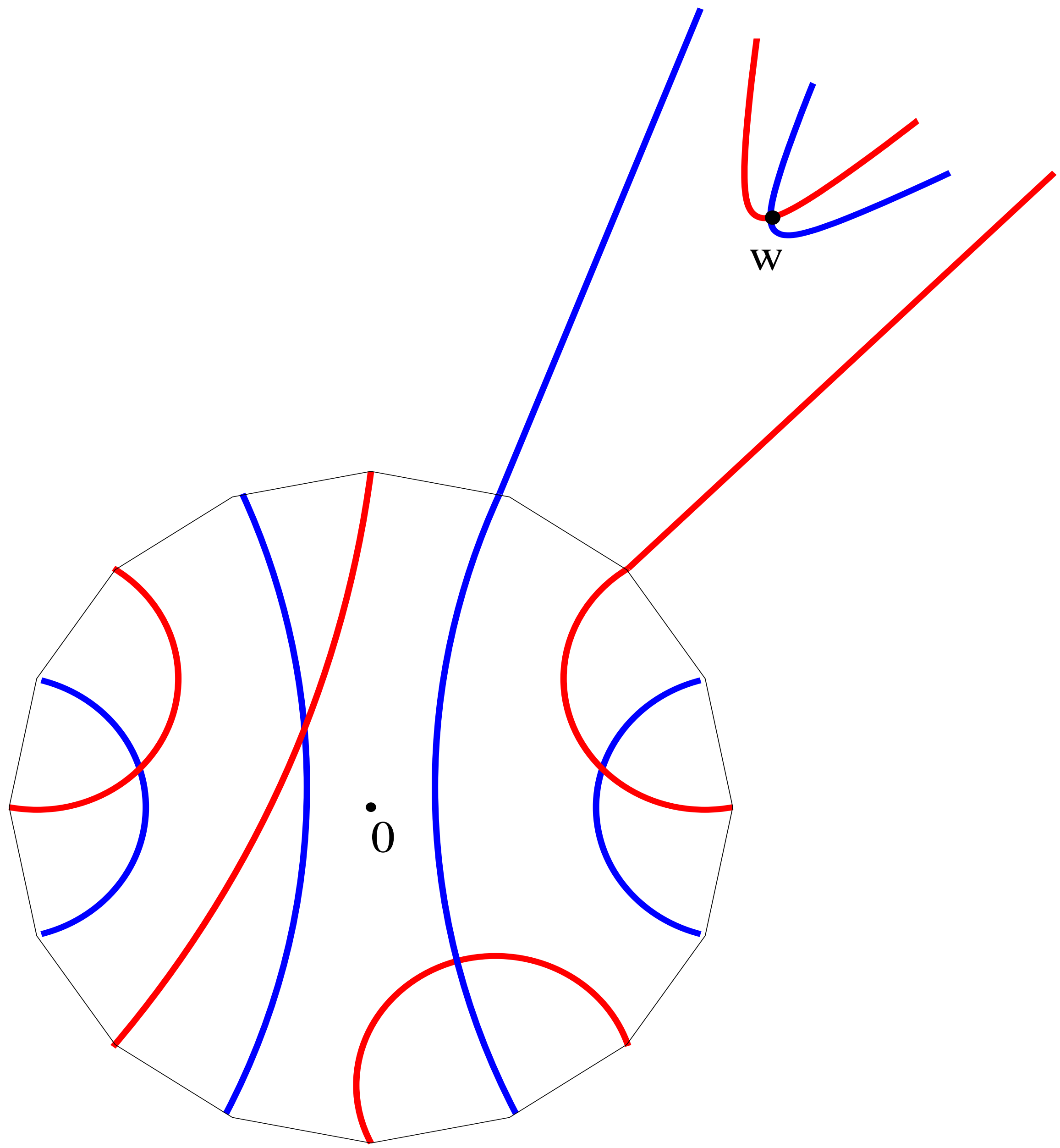


- Replace $f(z)$ with $g(z) = f(z)(z - w)$,

where $|w|$ is much greater than the magnitude of any root of f, and $\arg(w)$ is chosen so as to insert E in the right place.



- Confirm that the pre-existing branches are sufficiently unperturbed so that their topology remains the same.



Checking this requires tools from analysis and metric topology.

Necklaces

Let $f(z)$ be a polynomial of degree n with no repeated roots.

Let $z_1, \ldots, z_{n-1}$ be the zeroes of $f'(z)$, and suppose that

$$0 \le \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \pi,$$

where $\theta_i = \arg z_i$. Recall that the curve

$$C_\theta(f) \; = \; \{z \;\mid\; \operatorname{Im}(e^{-i\theta} f(z)) = 0\},$$

is nonsingular for $\theta \notin \{\theta_1, \ldots, \theta_{n-1}\}$.

- The **necklace of harmonic curves** of f is the family

$$\mathcal{C}_f = \{C_\theta(f) \;\mid\; \theta \in \mathbb{R}/\pi\mathbb{Z}\}.$$

The matching $M(C_\theta(f))$ is a constant M_i over each of the arcs

$$\mathcal{A}_1 = (\theta_1, \theta_2),\; \ldots,\; \mathcal{A}_{n-1} = (\theta_{n-1}, \theta_1).$$

- The **necklace of matchings** of f is

$$(M_1, \ldots, M_{n-1}),$$

regarded as a cyclically ordered $(n-1)$-tuple.

— For every i, the matchings M_i and M_{i+1} are related by a double transposition ("flip").

Similarly, we can define the **necklace of basketballs** of f.

How many necklaces of matchings are there?

For $n \leq 8$, there are $2(2n)^{n-2}$ necklaces of order n (EIS sequence A097629). This is the same as the number of unrooted directed trees on n vertices, but we haven't found a bijection.

What does the necklace tell you about the polynomial f, or about the location of its roots?

What if you start with a rational function f instead of a polynomial?

The curves $C_\theta(f)$ may contain loops, so the corresponding combinatorial object will be something more complicated than a noncrossing matching.