THE SLOPES DETERMINED BY n POINTS IN THE PLANE

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Abstract. Let m_{12}, m_{13}, ..., $m_{n-1,n}$ be the slopes of the $\binom{n}{2}$ lines connecting n points in general position in the plane. The ideal I_n of all algebraic relations among the m_{ij} defines a configuration space called the *slope variety of the complete graph*. We prove that I_n is reduced and Cohen-Macaulay, give an explicit Gröbner basis for it, and compute its Hilbert series combinatorially. We proceed chiefly by studying the associated Stanley-Reisner simplicial complex, which has an intricate recursive structure. In addition, we are able to answer many questions about the geometry of the slope variety by translating them into purely combinatorial problems concerning enumeration of trees.

1. Introduction

Let there be given $n \geq 2$ distinct points in the plane, connected in pairs by $\binom{n}{2}$ lines. The closure of the locus of all slope vectors $(m_{12}, \ldots, m_{n-1,n})$ arising from some such configuration is an irreducible algebraic variety of dimension $2n-3$, called the *affine slope variety of the complete graph* (as we soon explain). This slope variety turns out to have an unexpectedly rich combinatorial and geometric structure. Our techniques for investigating its properties draw on combinatorics (graph theory and recursive enumeration of trees), commutative algebra (Gröbner bases and Stanley-Reisner theory), and algebraic geometry. Before stating the main theorem, we set it in context by giving an overview of the theory of *graph varieties*, considered by the author in [10].

Let $\mathbb{P}^2$ be the projective plane over an algebraically closed field $\mathbf{k}$, and let G be a graph with vertices V and edges E. A *picture* $\mathbf{P}$ of G consists of a point $\mathbf{P}(v)$ for each vertex, and a line $\mathbf{P}(e)$ for each edge, subject to the conditions that $\mathbf{P}(v) \in \mathbf{P}(e)$ whenever v is an endpoint of e. Thus the data of n points and $\binom{n}{2}$ lines described earlier is a picture of the complete graph K_n on n vertices.

The set of all pictures of G is called the *picture space* $\mathcal{X}(G)$. A picture is *generic* if the points $\mathbf{P}(v)$ are all different; the closure of the locus of generic pictures is called the *picture variety* $\mathcal{V}(G)$. This is an irreducible component of $\mathcal{X}(G)$ of dimension $2|V|$. Passing to an affine open subset $\tilde{\mathcal{V}}(G) \subset \mathcal{V}(G)$, and projecting onto an affine space $\mathbb{A}_{\mathbf{k}}^{|E|}$ whose coordinates correspond to the slopes of lines $\mathbf{P}(e)$, we obtain the *affine slope variety* $\tilde{\mathcal{S}}(G)$, of dimension $2|V|-3$.

A *rigidity circuit* is a graph which admits a decomposition into two spanning trees, and contains no proper subgraph with that property. The most important rigidity circuits are the *wheels*: a wheel consists of a cycle with an attached central vertex. For each rigidity circuit C, there is a corresponding *tree polynomial* $\tau(C)$, a

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sum of signed squarefree monomials corresponding to spanning trees appearing in such decompositions; this polynomial is homogeneous and irreducible. The affine slope variety $\tilde{\mathcal{S}}(G)$ is cut out set-theoretically in $\mathbb{A}^{|E|}$ by the polynomials $\tau(C)$, where C ranges over all rigidity circuit subgraphs of G. These facts were proven in [10]. We can now state the main theorem of this paper.

Theorem 1.1. *Let $R_n = \mathbf{k}[m_{12}, \ldots, m_{n-1,n}]$, and let I_n be the ideal generated by the tree polynomials of all rigidity circuits in the complete graph K_n. Then:*

(i) *The affine slope variety $\tilde{\mathcal{S}}(K_n)$ is defined scheme-theoretically by I_n. That is, I_n is a prime ideal, and $\tilde{\mathcal{S}}(K_n) \cong \operatorname{Spec} R_n/I_n$.*

(ii) *The tree polynomials of the wheel subgraphs of K_n generate I_n, and form a Gröbner basis with respect to a certain graded lexicographic order.*

(iii) *$\tilde{\mathcal{S}}(K_n)$ has dimension $2n-3$ and degree*

$$\frac{(2n-4)!}{2^{n-2}(n-2)!} = (2n-5)(2n-7)\cdots(3)(1), \tag{1}$$

the number of perfect matchings on $[1, 2n-4] = \{1, 2, \ldots, 2n-4\}$. Furthermore, the Hilbert series of R_n/I_n is

$$\frac{\sum_{k=0}^{n-2} h(n,k)t^k}{(1-t)^{2n-3}},$$

where $h(n,k)$ counts the number of perfect matchings on $[1, 2n-4]$ with exactly k long pairs*, that is, pairs not of the form $\{i, i+1\}$.*

(iv) *The ring R_n/I_n and the affine slope variety $\tilde{\mathcal{S}}(K_n)$ are Cohen-Macaulay.*

We begin in Section 2 by describing the basic objects—graph varieties and tree polynomials—in somewhat more detail. We do this both to make this paper more self-contained, and because several details of the constructions will be of importance later on. The reader is referred to [10] and [9] for more leisurely treatments of these subjects.

In the first main part of the paper, we construct a monomial ideal J_n, generated by the initial terms of tree polynomials $\tau(W)$ of wheels $W \subset K_n$ with respect to a certain graded lexicographic term ordering. As mentioned previously, the monomials of $\tau(W)$ correspond to *coupled* spanning trees of W, that is, whose complements are also spanning trees. In order to identify the leading term of a wheel polynomial, we need several technical facts about the valences of vertices in coupled trees; these facts comprise Section 3. In Section 4, we introduce the term ordering and, in Theorem 4.3, give a necessary and sufficient combinatorial condition on monomials (somewhat akin to pattern avoidance in permutations) for membership in J_n.

The second part of the paper consists of Sections 5–9. Here we study the Stanley-Reisner simplicial complex $\Delta(n)$ whose faces correspond to squarefree monomials that do not belong to J_n. This simplicial complex has a surprising amount of combinatorial structure. First, $\Delta(n)$ is pure, and all its facets may be built up recursively from facets of smaller Stanley-Reisner complexes. Second, this recurrence may be translated into a bijection between facets and a combinatorially more natural set, the *binary total partitions*, which are enumerated by the double factorial numbers (1). These numbers therefore give the degree (or multiplicity) of the ideal J_n. Third, the description of facets leads to a proof that $\Delta(n)$ is shellable and hence Cohen-Macaulay. This part of the paper (Section 8) is very technical; the arguments are combinatorially elementary but do require careful bookkeeping. The shelling argument leads in turn to a recursive computation of the h-vector of $\Delta(n)$; the coefficients of the h-vector enumerate perfect matchings by the number of long pairs (a combinatorial problem first considered by Kreweras and Poupard [8]).

At this point, we do not yet know that these results on $\Delta(n)$ correspond to properties of the affine slope variety. The missing piece is to show that J_n is not too small—precisely, that it is the initial ideal of an ideal defining $\tilde{\mathcal{S}}(K_n)$ scheme-theoretically. As it turns out, it is enough to show that the double factorial numbers give a lower bound for the degree of $\tilde{\mathcal{S}}(K_n)$. We prove this in Sections 10 and 11. Our approach is to consider a nested family of algebraic subsets of $\tilde{\mathcal{S}}(K_n)$ called *flattened slope varieties*, whose degree can be bounded from below by a recursive formula (Theorem 10.4). We show that this recurrence is equivalent to one enumerating combinatorial objects called *decreasing planar trees*, which, like matchings and binary total partitions, are enumerated the double factorials. Using the fact that J_n is Cohen-Macaulay and has the appropriate codimension and degree, we conclude that J_n is the (complete) initial ideal of I_n under an appropriate term ordering; equivalently, the wheel polynomials form a Gröbner basis. The assertions of the main theorem follow more or less immediately.

Together with [10], these results constitute the author's doctoral dissertation [9]. The author thanks his thesis advisor, Mark Haiman, for his ongoing support.

2. Preliminaries: Graphs and Tree Polynomials

We assume that the reader is familiar with the elements of graph theory, for which a good general reference is [15]. We first fix some notation and terminology. The symbol $\mathbb{N}$ denotes the positive integers. We abbreviate the set $\{m, m+1, \ldots, n\}$ by $[m, n]$.

A *graph* G is a pair (V, E), where $V = V(G)$ is a finite set of *vertices* and $E = E(G)$ is a set of *edges*, or unordered pairs of distinct vertices $e = \{v, w\}$. (Thus we do not allow loops or multiple edges.) For ease of use, we frequently abbreviate $\{v, w\}$ by vw. The vertices v, w are the *endpoints* of e. A *subgraph* of G is a graph $G' = (V', E')$ with $V' \subset V$ and $E' \subset E$. We use the symbols $+$ and $-$ to denote addition and deletion of edges.

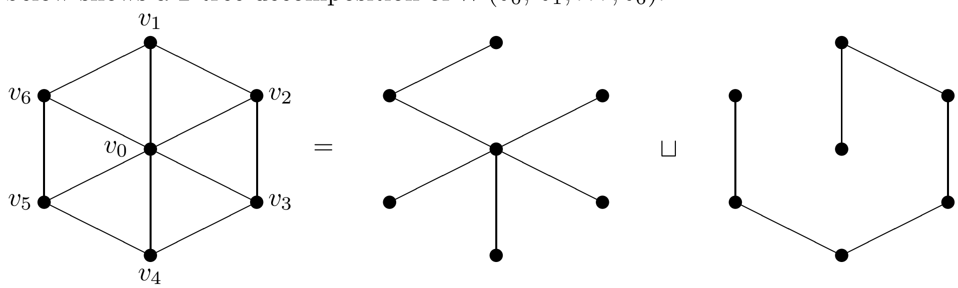
The *valence* of v with respect to an edge set E, written $\mathrm{val}_E(v)$, is the number of edges in E incident to v. (This is more usually called the *degree*, but we wish to reserve that term for a different usage.) A vertex of valence 1 is called a *leaf*. The *support* of an edge set E is $V(E) = \{v \mid \mathrm{val}_E(v) > 0\}$. The *complete graph* K_V is the graph with vertex set V and every two vertices adjacent; thus $|E(K_V)| = \binom{|V|}{2}$. We abbreviate $K_{[1,n]}$ by K_n. For convenience, we frequently ignore the technical distinction between an edge set E and the graph $(V(E), E)$.

Let $v_1, \ldots, v_k$ be distinct vertices. The edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ is called a *path* from v_1 to v_k, and if $k \geq 3$ then the edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$ is called a *cycle* or *k-cycle*. It is frequently convenient to describe a path or cycle by listing its vertices in order.

A graph G is *connected* if every pair of vertices belongs to some common path; it is a *tree* if it is connected and contains no cycle. Equivalently, a tree may be defined as a connected graph with $|E(G)| = |V(G)| - 1$, or as a graph in which every pair of vertices belongs to exactly one common path. A *spanning tree* of G is a tree T with $V(T) = V(G)$ and $E(T) \subset E(G)$. The *connected components* of a graph are its maximal connected subgraphs.

A graph $G = (V, E)$ is a *rigidity pseudocircuit* if $E = T \sqcup T'$, where T, T' are spanning trees and the symbol $\sqcup$ denotes a disjoint union. A rigidity pseudocircuit is a *rigidity circuit* if it contains no other rigidity pseudocircuit as a proper subgraph. (For the reasons behind this terminology, see [7].) A spanning tree $T \subset E$ is called *coupled* if its complement is also a spanning tree; we denote the set of coupled trees of G by $\mathrm{Cpl}(G)$.

Let $v_0, v_1, \ldots, v_k$ be distinct vertices. The *k-wheel* $W = W(v_0;\, v_1, \ldots, v_k)$ is defined as the graph with edges $\{v_0v_1, v_1v_2, \ldots, v_0v_k\} \cup \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$. It is easy to check that every wheel is a rigidity circuit [7, Exercise 4.13]; the figure below shows a 2-tree decomposition of $W(v_0;\, v_1, \ldots, v_6)$.



The vertex v_0 is called the *center* of W, and the vertices $v_1, \ldots, v_k$ are its *spokes*. An edge joining two spokes is a *chord* of the wheel; an edge joining a spoke to the center is a *radius*. We denote the sets of chords and radii by $\mathrm{Ch}(W)$ and $\mathrm{Rd}(W)$ respectively. It is notationally convenient to set $v_{k+1} = v_1$, so that $\mathrm{Ch}(W) = \{v_iv_{i+1} \mid i \in [1, k]\}$.

We now describe certain polynomials associated to rigidity circuits. Let $n \geq 2$ be an integer and $\mathbf{k}$ an algebraically closed field. We will work over the polynomial ring in $\binom{n}{2}$ variables

$$R_n = \mathbf{k}[m_{12},\, \ldots,\, m_{n-1,n}].$$

To each edge set $E \subset E(K_n)$ we associate the squarefree monomial

$$m_E = \prod_{e \in E} m_e. \tag{2}$$

For every rigidity circuit C in K_n, there is an irreducible polynomial, the *tree polynomial* of C, defined up to sign and having the form

$$\tau(C) = \sum_{T \in \mathrm{Cpl}(C)} \varepsilon(T) m_T, \tag{3}$$

where $\varepsilon(T) \in \{1, -1\}$. By [10, Theorem 5.4], the tree polynomial is irreducible and homogeneous of degree $|V(C)| - 1 = |E(C)|/2$.

We summarize the construction of $\tau(C)$; for more details and examples, see [10]. For $1 \leq i < j \leq n$, regard the edge $\{i, j\}$ as an oriented edge (i, j), and formally set $(j, i) = -(i, j)$. Fix a spanning tree $T \subset E(C)$ (not necessarily coupled), and let $\overline{T} = E(C) \backslash T$. For every edge $e = vw \in \overline{T}$, the edge set $T + e$ contains a unique cycle $P_T(e)$, which we may regard as a set of oriented edges $\{(v, w),\, (w, w_1),\, \ldots,\, (w_r, v)\}$.

Let M be the $(|V|-1)\times(|V|-1)$ square matrix with rows indexed by edges $e\in\overline{T}$ and columns indexed by the edges $f\in T$, and whose (e,f) entry is

$$M_{e,f}=\begin{cases} m_e-m_f & \text{if} \quad f\in P_T(e),\\ m_f-m_e & \text{if} \ -f\in P_T(e),\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

The tree polynomial is then defined as $\tau(C)=\det M$. Up to sign, this construction is independent of the choice of the tree T.

The *affine slope variety* $\tilde{\mathcal{S}}(n)=\tilde{\mathcal{S}}(K_n)$ is defined as follows. Let $p_1,\dots,p_n$ be n distinct points in the affine plane $\mathbb{A}^2_{\mathbf{k}}$, with no two points lying on the same vertical line. Let $m_{ij}\in\mathbf{k}$ be the slope of the unique line joining p_i and p_j. Thus $(m_{12},\dots,m_{n-1,n})$ is a point in affine $\binom{n}{2}$-space $\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}R_n$, and $\tilde{\mathcal{S}}(n)$ is defined as the closure of the locus of all such points arising from the data $(p_1,\dots,p_n)$.

By [10, Theorem 5.6], the ideal generated by the tree polynomials

$$I_n=\big\langle\tau(C)\ \mid\ C\subset K_n \text{ is a rigidity circuit}\big\rangle \tag{5}$$

cuts out the affine slope variety $\tilde{\mathcal{S}}(n)$ inside $\operatorname{Spec}R_n$. In fact, I_n is generated by the tree polynomials of wheel subgraphs of K_n. We omit the proof of this fact, since we shall eventually prove the following stronger result: the wheel polynomials form a Gröbner basis for I_n with respect to any of a large class of term orderings.

3. Vertex Valences in Coupled Trees

This section contains several technical facts concerning the valences of vertices in couples spanning trees of a wheel. These observations lead eventually to an explicit identification of the initial monomial of a wheel polynomial $\tau(W)$ with respect to a certain term ordering.

Throughout this section, we fix a k-wheel $W=W(v_0;v_1,\dots,v_k)\subset E(K_n)$. If $T\subset E(W)$ is a coupled spanning tree of W, we set $\overline{T}:=E(W)\setminus T$.

For each $i\in[1,k]$ and each coupled tree $T\in\operatorname{Cpl}(W)$, either $\operatorname{val}_T(v_i)=1$ or $\operatorname{val}_T(v_i)=2$. In addition, not all spokes v_i have the same valence, since T is neither $\operatorname{Ch}(W)$ nor $\operatorname{Rd}(W)$. Thus val_T may be regarded as a nonconstant function from $[1,k]$ to $[1,2]$.

Lemma 3.1. *Let $T\in\operatorname{Cpl}(W)$ and $i,j\in[1,k]$. Then $\overline{T}$ contains at least one of the following four edges: v_0v_i, v_0v_j, v_iv_{i+1}, and $v_{j-1}v_j$.*

Proof. Suppose not. Let i,j be a counterexample such that $j-i$ is as small as possible. If necessary, we may reindex the spokes so that $i\le j$. If $j=i$, then $\operatorname{val}_T(v_i)=3$, which is impossible. If $j-i=1$, then T contains the cycle v_0,v_i,v_j,v_0, and if $j-i=2$, then T contains the cycle v_0,v_i,v_{i+1},v_j,v_0.

Now suppose $j-i>2$. Since T contains the path $v_{i+1},v_i,v_0,v_j,v_{j-1}$, it cannot contain the path $v_{i+1},v_{i+2},\dots,v_{j-1}$. Let k and ℓ be the least and greatest indices, respectively, such that $v_kv_{k+1}\notin T$, $v_{\ell-1}v_\ell\notin T$, and $i<k\le\ell<j$. Now $v_0v_k\notin T$, otherwise T contains the cycle $v_0,v_i,v_{i+1},\dots,v_k,v_0$. For a similar reason, $v_0v_\ell\notin T$. But then k,ℓ is a counterexample to the lemma, and $|\ell-k|<|j-i|$, which contradicts the choice of i and j. □

Lemma 3.2. *For each $T \in \mathrm{Cpl}(W)$, at least one of the following conditions is true: either*

$$\text{for all } i \in [1,k], \quad v_0v_i \in T \ \text{ if and only if } \ v_iv_{i+1} \in \overline{T},$$

or

$$\text{for all } i \in [1,k], \quad v_0v_i \in T \ \text{ if and only if } \ v_{i-1}v_i \in \overline{T}.$$

Proof. Suppose that both conditions fail. That is, there exists $i \in [1,k]$ such that either $v_0v_i, v_iv_{i+1} \in T$ or v_0v_i, $v_iv_{i+1} \in \overline{T}$. Interchanging T and $\overline{T}$ if necessary, we may assume the former. Moreover, there exists $j \in [1,k]$ such that either v_0v_j, $v_{j-1}v_j \in T$ or v_0v_j, $v_{j-1}v_j \in \overline{T}$. The former is ruled out by Lemma 3.1, so the latter must hold; in particular $i \neq j$.

Since $v_0v_i, v_iv_{i+1} \in T$, the edge v_0v_{i+1} must belong to $\overline{T}$. In particular $\overline{T}$ contains the path $v_{j-1}, v_j, v_0, v_{i+1}$. Thus $\overline{T}$ does not contain the path $v_{i+1}, v_{i+2}, \dots, v_{j-1}$. Let h be the largest number in $[i+1, j-1]$ such that $v_{h-1}v_h \in T$. Since the path $v_h, v_{h+1}, \dots, v_j, v_0$ is contained in $\overline{T}$, the edge v_0v_h must belong to T. It follows that T contains the edges v_0v_i, v_iv_{i+1}, $v_{h-1}v_h$, and v_0v_h, contradicting Lemma 3.1. □

Proposition 3.3. *Let $d : [1,k] \to [1,2]$ be a nonconstant function. Then there exist exactly two coupled trees of W for which $\mathrm{val}_T = d$.*

Proof. To simplify the notation, write $m_{i,j}$ for $m_{v_iv_j}$. Also, we put $v_{k+1} = v_1$. Putting $T = \mathrm{Rd}(W)$ in (4), the matrix M becomes

$$\begin{bmatrix} m_{0,1} - m_{1,2} & m_{1,2} - m_{0,2} & 0 & \dots & 0 \\ 0 & m_{0,2} - m_{2,3} & m_{2,3} - m_{0,3} & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ m_{0,k} - m_{k,1} & 0 & \dots & \dots & m_{k,1} - m_{0,1} \end{bmatrix}$$

Taking the determinant, we obtain

$$\begin{aligned} \tau(W) &= \prod_{i=1}^{k}(m_{0,i} - m_{i,i+1}) \ + \ (-1)^{k-1}\prod_{i=1}^{k}(m_{i,i+1} - m_{0,i+1}) \\ &= \prod_{i=1}^{k}(m_{0,i} - m_{i,i+1}) \ - \ \prod_{i=1}^{k}(m_{0,i+1} - m_{i,i+1}). \end{aligned} \tag{6}$$

The monomial $m_{\mathrm{Ch}(W)}$ appears in both products in (6), once with coefficient $+1$ and once with -1. The same is true for the monomial $m_{\mathrm{Rd}(W)}$. One may easily verify that no other cancellation occurs. Accordingly, to enumerate the number of coupled trees by the valences of spokes, we may substitute z_iz_{i+1} for $m_{i,i+1}$ and z_i for $m_{0,i}$ in (6) (where the z_i are indeterminates) and change all the $-$'s to $+$'s. This yields the expression

$$\prod_{i=1}^{k}(z_i + z_iz_{i+1}) \ + \ \prod_{i=1}^{k}(z_{i+1} + z_iz_{i+1}) \ - \ 2\left(\prod_{i=1}^{k} z_i + \prod_{i=1}^{k} z_i^2\right) \ = \ 2\sum_{d} z_i^{d(i)},$$

where the sum is taken over all nonconstant functions $d : [1,k] \to [1,2]$. □

Remark 3.4. Proposition 3.3 has the following consequence, which may also be obtained by direct counting: every k-wheel has exactly $2^{k+1} - 4$ coupled spanning trees.

Definition 3.5. Let $d : [1, k] \to [1, 2]$ be a nonconstant function. The *type* of a chord $v_i v_{i+1}$ with respect to d is the pair of numbers $d(i), d(i+1)$. The *type* of a radius $v_0 v_i$ is the pair $d(i-1), d(i+1)$. If $T \in \mathrm{Cpl}(W)$, we define the type of an edge with respect to T to be its type with respect to $d = \mathrm{val}_T$. For brevity, we will speak of *type*-11 *chords*, *type*-12 *radii*, etc.

Lemma 3.6. *Let $T \in \mathrm{Cpl}(W)$, and define the types of chords and radii of W with respect to the function val_T. Then:*

(i) *Every type-22 chord belongs to T.*
(ii) *Every type-11 chord belongs to $\overline{T}$.*
(iii) *Every type-22 radius belongs to $\overline{T}$.*
(iv) *Every type-11 radius belongs to T.*

Proof. Let $v_i v_{i+1}$ be a chord. If (i) fails, then the edges $v_{i-1}v_i$, $v_{i+1}v_{i+2}$, $v_0 v_i$, $v_0 v_{i+1}$ all belong to T. If (ii) fails, then those edges all belong to $\overline{T}$. In either case, Lemma 3.1 is contradicted.

Now let $v_0 v_i$ be a radius Statements (iii) and (iv) are equivalent (switch T and $\overline{T}$), so we prove only (iii). If $\mathrm{val}_T(v_i) = 2$, then T contains the chords $v_i v_{i+1}$ and $v_{i-1}v_i$ by parts (i) and (ii) of the lemma, so the radius $v_0 v_i$ belongs to $\overline{T}$. On the other hand, if $\mathrm{val}_T(v_i) = 1$ and $v_0 v_i \in T$, then $\overline{T}$ contains the chords $v_i v_{i+1}$ and $v_{i-1}v_i$. Since $\mathrm{val}_T(v_{i+1}) = \mathrm{val}_T(v_{i-1}) = 2$ by hypothesis, the edges $v_{i-2}v_{i-1}$, $v_{i+1}v_{i+2}$, $v_0 v_{i-1}$, $v_0 v_{i+1}$ must all belong to T, which contradicts Lemma 3.1. □

Lemma 3.7. *Let $T \in \mathrm{Cpl}(W)$ and $1 \le i < j \le k$. Suppose that either*

$$\mathrm{val}_T(v_i) = 1, \qquad \mathrm{val}_T(v_{i+1}) = \cdots = \mathrm{val}_T(v_j) = 2, \qquad \mathrm{val}_T(v_{j+1}) = 1 \tag{7a}$$

or

$$\mathrm{val}_T(v_i) = 2, \qquad \mathrm{val}_T(v_{i+1}) = \cdots = \mathrm{val}_T(v_j) = 1, \qquad \mathrm{val}_T(v_{j+1}) = 2. \tag{7b}$$

Then exactly one of the two chords $v_i v_{i+1}$, $v_j v_{j+1}$ belongs to T.

Proof. Suppose that (7a) holds. Then the chords $v_{i+1}v_{i+2}$, $v_{i+2}v_{i+3}$, $\ldots$, $v_{j-1}v_j$ all belong to T. If both $v_i v_{i+1}$ and $v_j v_{j+1}$ belong to T, then the path $v_i, v_{i+1}, \ldots, v_{j+1}$ is a connected component of T, which is impossible. On the other hand, if neither of those chords belong to T, then $v_0 v_{i+1}$ and $v_0 v_j$ both belong to T. But then T contains the cycle $v_0, v_{i+1}, v_{i+2}, \ldots, v_j, v_0$, which is impossible, If we assume (7b) instead of (7a), the same argument goes through, switching 2 with 1 and T with $\overline{T}$. □

An alternate formulation of this lemma is as follows. Let a nonconstant function $d : [1, k] \to [1, 2]$ be given, and let T be a coupled tree with $\mathrm{val}_T = d$. Traverse the chords of W in order, coloring the type-12 chords (of which there are a positive even number) alternately red and blue. Then either the red chords all belong to T and the blue chords all belong to $\overline{T}$, or vice versa. Moreover, choosing the color of a single type-12 chord suffices to determine the rest. Having made such a choice, a radius $v_0 v_i$ belongs to T exactly when $d(i) - |T \cap \{v_{i-1}v_i,\, v_i v_{i+1}\}| = 1$.

Alternatively, if the function $d = \mathrm{val}_T$ is given, then to determine T uniquely it suffices to specify whether a single type-12 radius $v_0 v_i$ belongs to T or to $\overline{T}$. Without loss of generality $v_{i-1}v_i$ is of type 11 or 22, and $v_i v_{i+1}$ is of type 12. The value of $d(i)$ determines whether or not $v_i v_{i+1}$ belongs to T, so the rest of T is determined uniquely as in the preceding paragraph. That is:

Proposition 3.8. *Let $T \in \mathrm{Cpl}(W)$. Define the type of each edge in W with respect to val_T. Then T contains all type-22 chords, all type-11 radii, half the type-12 chords, in alternation, and a corresponding half of the type-12 radii.*

We conclude our technical preliminaries with two results describing the conditions under which a pair of complementary spanning trees may swap edges.

Lemma 3.9. *Let $T \in \mathrm{Cpl}(W)$. Suppose that $v_{i-1}v_i$ and v_iv_{i+1} belong to $\overline{T}$, so that $v_0v_i \in T$. Assume without loss of generality that the path in $\overline{T}$ from v_i to v_0 passes through v_{i+1}. Then W admits the 2-tree decompositions*

$$E_1 = T - v_0v_i + v_iv_{i+1}, \qquad E_2 = \overline{T} - v_iv_{i+1} + v_0v_i$$

and

$$F_1 = T - v_0v_{i-1} + v_{i-1}v_i, \qquad F_2 = \overline{T} - v_{i-1}v_i + v_0v_{i-1}.$$

Proof. Clearly $E_1 \sqcup E_2 = F_1 \sqcup F_2 = W$. The edge set E_1 is a tree because v_i is a leaf of T, and E_2 is a tree because v_i and v_0 are in different connected components of $\overline{T} - v_iv_{i+1}$. Meanwhile, F_1 is a tree because v_i and v_{i-1} are in different connected components of $T - v_0v_{i-1}$, and F_2 is a tree because v_{i-1} and v_0 are in different connected components of $\overline{T} - v_{i-1}v_i$. □

Lemma 3.10. *Let $T \in \mathrm{Cpl}(W)$. Suppose that $v_{i-1}v_i \in \overline{T}$ and that v_0v_i, $v_iv_{i+1} \in T$, so that $v_0v_{i+1} \in \overline{T}$. Then:*

(i) *If T contains at least one radius other than v_0v_i, then W admits the 2-tree decomposition*

$$E_1 = T - v_0v_i + v_{i-1}v_i, \qquad E_2 = \overline{T} - v_{i-1}v_i + v_0v_i.$$

(ii) *If $\overline{T}$ contains at least one radius other than v_0v_{i+1}, then W admits the 2-tree decomposition*

$$F_1 = T - v_iv_{i+1} + v_0v_{i+1}, \qquad F_2 = \overline{T} - v_0v_{i+1} + v_iv_{i+1}.$$

Proof. Clearly W is the disjoint union of E_1 and E_2 (resp. F_1 and F_2), so it suffices to show that these edge sets are in fact trees.

(i) E_2 is a tree because v_i is a leaf of $\overline{T}$. If the path from v_{i-1} to v_i in T does not go through v_0, then it must be $\mathrm{Ch}(W) - v_{i-1}v_i$. But then T contains at least $k-1$ chords and two radii, which is impossible. Therefore v_{i-1} and v_i lie in different connected components of $T - v_0v_i$, and E_1 is a tree.

(ii) The path from v_0 to v_{i+1} in T is just v_0, v_i, v_{i+1}, so v_0 and v_{i+1} belong to different connected components of $T - v_iv_{i+1}$. Hence F_1 is a tree. If the path from v_i to v_{i+1} in $\overline{T}$ does not go through v_0, then it must be $\mathrm{Ch}(W) - v_iv_{i+1}$, which is impossible. So v_i and v_{i+1} are in different connected components of $\overline{T} - v_0v_{i+1}$. Hence F_2 is a tree. □

4. The Leading Tree of a Wheel

The main result of this section is Theorem 4.3, in which we describe explicitly the ideal generated by the initial terms of wheel polynomials. Fix once and for all the following *lexicographic* order $>$ on the variables m_{ij}:

$$m_{12} > m_{13} > \ldots > m_{1n} > m_{23} > \ldots$$

The corresponding total order for edges of K_n is

$$12 > 13 > \ldots > 1n > 23 > \ldots \tag{8}$$

We next extend $>$ to a term ordering on R_n, **graded lexicographic order**, which is defined as follows: $\prod_{i,j} m_{ij}^{a_{ij}} > \prod_{i,j} m_{ij}^{b_{ij}}$ if either

$$\begin{aligned} \sum a_{ij} \;&>\; \sum b_{ij}, \qquad \text{or} \\ \sum a_{ij} \;&=\; \sum b_{ij} \quad \text{and} \quad a_{k\ell} \;>\; b_{k\ell}, \end{aligned} \tag{9}$$

where $m_{k\ell}$ is the greatest variable (in lexicographic order) such that $a_{k\ell} \neq b_{k\ell}$.

Associating edge sets with square-free monomials as in (2), we may regard the term ordering on R_n as defining an extension of the ordering on edges (8) to a total order on subsets of $E(K_n)$. Then (9) becomes the following: for $E, F \subset E(K_n)$, $E > F$ if either $|E| > |F|$, or else $|E| = |F|$ and $\max(E\#F) \in E$, where the symbol $\#$ denotes the symmetric difference operator.

Given a wheel $W \subset E(K_n)$, we wish to identify the *leading tree* $LT(W)$ of W, that is, the coupled tree of W corresponding to the leading monomial of $\tau(W)$ (with respect to graded lex order). We begin by computing the valence of each spoke of $LT(W)$, using the tools developed in the previous section. By Proposition 3.8, this will rule out all but two possibilities for the leading tree.

Proposition 4.1. *Let $W = W(v_0; v_1, \ldots, v_k)$, with $V = V(W) \subset [1, n]$. Then:*

(i) *Suppose that $v_0 = \min(V)$. Then $\mathrm{val}_{LT(W)}(v_0) = k - 1$.*
(ii) *Suppose that $v_0 = \max(V)$. Then $\mathrm{val}_{LT(W)}(v_0) = 1$.*
(iii) *Suppose that $v_0 \notin \{\min(V), \max(V)\}$. Then for all $i \in [1, k]$,*

$$\mathrm{val}_{LT(W)}(v_i) = \begin{cases} 1 & \text{if } v_i > v_0, \\ 2 & \text{if } v_i < v_0. \end{cases}$$

In particular, if v_0 is the rth largest member of V, where $r \in [2, k]$, then

$$\mathrm{val}_{LT(W)}(v_0) = r - 1.$$

Proof. (i,ii) Suppose $v_0 = \min(V)$. We will show that if T is a coupled tree with $\mathrm{val}_T(v_0) < k - 1$, then T cannot be the leading tree of W. Note that $\overline{T}$ contains at least two radii, say v_0v_i and v_0v_j. At least one of the chords $v_{i-1}v_i$, v_iv_{i+1} belongs to T. If both do, then by Lemma 3.9, at least one of

$$T_1 = T - v_{i-1}v_i + v_0v_i, \qquad T_2 = T - v_iv_{i+1} + v_0v_i$$

is coupled. If $v_{i-1}v_i \in T$ and $v_iv_{i+1} \in \overline{T}$, then T_1 is coupled by Lemma 3.10 (i). But $T_1 > T$, so $T \neq LT(W)$ as desired. The proof of (ii) is analogous.

(iii) Suppose that v_0 is neither the minimum nor the maximum element of V. Let T be a coupled tree of W such that $\mathrm{val}_T(v_i) = 1$ for some $v_i < v_0$. To show that $T \neq LT(W)$, we will construct a tree $T' \in \mathrm{Cpl}(W)$ with $T' > T$. There are two cases to consider.

Case 1: $v_0v_i \in T$. Then $\overline{T}$ contains the chords $v_{i-1}v_i$ and v_iv_{i+1}. Without loss of generality, we may assume that the path from v_i to v_0 in $\overline{T}$ passes through v_{i+1}. Then $v_0v_{i-1} \in T$. By Lemma 3.9, the tree $T' = T - v_0v_{i-1} + v_{i-1}v_i$ is coupled, and $v_{i-1}v_i > v_0v_{i-1}$, so $T' > T$.

Case 2: $v_0v_i \in \overline{T}$. Without loss of generality, $v_iv_{i+1} \in \overline{T}$ and $v_{i-1}v_i \in T$, so $v_0v_{i+1} \in T$. Let $T' = T - v_0v_{i+1} + v_iv_{i+1}$. Note that $T' > T$. If T' is coupled, then we are done. Otherwise, Lemma 3.10 (i) implies that v_0v_{i+1} is the unique radius in T, that is, T is the path v_0, v_{i+1}, v_i, $\ldots$, v_{i-2}, v_{i-1}. Since $v_0 \neq \max(T)$, we may choose j such that $v_j > v_0$; note that $j \neq i$. Then the tree $T' = T - v_j v_{j+1} + v_0 v_{j+1}$ is coupled, and $T' > T$. $\square$

Proposition 4.2. *Let $W \subset E(K_n)$ be a k-wheel with vertices $V = V(W)$ and center v_0.*

(i) *Suppose $v_0 = \min(V)$. Label the spokes so that $W = W(v_0; v_1, \dots, v_k)$ with $v_1 = \max\{v_1, \dots, v_k\}$ and $v_2 > v_k$. Then*

$$LT(W) = \mathrm{Rd}(W) - v_0 v_1 + v_k v_{k+1}.$$

(ii) *Suppose $v_0 = \max(V)$. Label the spokes so that $W = W(v_0; v_1, \dots, v_k)$ with $v_1 v_2 = \min(\mathrm{Ch}(W))$ and $v_1 > v_2$. Then*

$$LT(W) = \mathrm{Ch}(W) - v_1 v_2 + v_0 v_2.$$

Proof. (i) By Proposition 4.1 (i), $LT(W)$ contains exactly one chord. Since $v_0 v_i > v_j v_{j+1}$ for all i, j, the unique radius not in $LT(W)$ must be $\min(\mathrm{Rd}(W)) = v_0 v_1$. This implies the desired result because $v_k v_{k+1} > v_1 v_2$.

(ii) For each $1 \in [1, k]$, define

$$T_i = \mathrm{Ch}(W) + v_0 v_i - \min\left(v_{i-1} v_i,\ v_i v_{i+1}\right).$$

By Proposition 4.1 (ii), $LT(W)$ contains exactly one radius, so $LT(W) = T_i$ for some i. Note that $T_i = \mathrm{Ch}(W) + v_0 v_i - v_1 v_2$ for $i = 1, 2$; in particular $\max(T_1 \# T_2) = v_0 v_2 \in T_2$. On the other hand, for $i > 2$, we have

$$\begin{aligned} \max(T_i \# T_2) &= \max\left(v_0 v_i,\ v_0 v_2,\ v_1 v_2,\ \min(v_{i-1} v_i, v_i v_{i+1})\right) \\ &= \min(v_{i-1} v_i,\ v_i v_{i+1}) \in T_2. \end{aligned}$$

We conclude that $LT(W) = T_2$. $\square$

In the case that $v_0 \notin \{\min(V), \max(V)\}$, we have by Proposition 4.1 (iii)

$$\mathrm{val}_{LT(W)}(i) = \begin{cases} 1 & \text{if } v_i > v_0 \\ 2 & \text{if } v_i < v_0. \end{cases}$$

By Proposition 3.3, there are exactly two coupled trees $T, T' \in \mathrm{Cpl}(W)$ satisfying these conditions. Moreover

$$\begin{aligned} T \cap T' &= \{\text{chords of type } 22\} \cup \{\text{radii of type } 11\}, \\ T \# T' &= \{\text{chords of type } 12\} \cup \{\text{radii of type } 12\}. \end{aligned}$$

Define the *critical edge* of W to be the maximum element of $T \# T'$. Thus $LT(W)$ is whichever of T, T' contains the critical edge.

Theorem 4.3. *Let T be a tree with $V(T) \subset [1, n]$. Then the following are equivalent:*

(i) *There exists a wheel $W \subset K_n$ such that $T = LT(W)$.*

(ii) *T contains a path $(v_1, \dots, v_k)$ satisfying the conditions*

$$\begin{aligned} &k \geq 4, \\ &\max(v_1, \dots, v_k) = v_1, \\ &\max(v_2, \dots, v_k) = v_k, \\ &v_2 > v_{k-1}. \end{aligned} \tag{10}$$

Proof. The easier direction is (ii) $\implies$ (i). Suppose that the path $P = (v_1, \dots, v_k)$ satisfies (10). Consider the wheel $W = W(v_k; v_1, v_2, \dots, v_{k-1})$. By Proposition 4.1, $LT(W)$ is a path from v_k to v_1. The two possibilities for $LT(W)$ are P and the path

$$P' = (v_k,\, v_2,\, v_3,\, \dots,\, v_{k-2},\, v_{k-1},\, v_1).$$

Since $v_1 > v_k > v_2 > v_{k-1}$, the critical edge of W is

$$\begin{aligned} \max(P\#P') &= \max(v_1v_2,\, v_{k-1}v_k,\, v_kv_2,\, v_{k-1}v_1) \\ &= v_{k-1}v_k \in P \end{aligned}$$

so $P = LT(W)$, as desired.

We now show that (i) $\implies$ (ii). Let $W = W(v_0; v_1, \dots, v_k)$ with $LT(W) = T$. We will show that T contains a path P satisfying (10).

Case 1: $v_0 = \min(V)$. Reindex the spokes of W so that $v_1 = \max(v_1, \dots, v_k)$ and $v_2 > v_k$. By Proposition 4.2 (i) we have

$$LT(W) = \mathrm{Rd}(W) - v_0v_1 + v_kv_{k+1}.$$

Since $v_0 < v_k < v_2 < v_1$, we may take P to be the path $(v_1,\, v_k,\, v_0,\, v_2)$.

Case 2: $v_0 = \max(V)$. Reindex the spokes of W so that $v_1v_2 = \min(\mathrm{Ch}(W))$ and $v_1 > v_2$. By Proposition 4.2 (ii) we have

$$LT(W) = \mathrm{Ch}(W) - v_1v_2 + v_0v_2.$$

Let $v_j = \max(v_1, \dots, v_k)$. Obviously $j \neq 2$. Also $j \neq 3$ (since $v_1v_2 < v_2v_3$, implying $v_1 > v_3$). Consider the path $P = (v_0,\ v_2,\, v_3,\, \dots,\, v_j)$. The two largest vertices of P are v_0 and v_j. If $v_2 < v_{j-1}$ then $v_{j-1}v_j < v_1v_2$, a contradiction. So P satisfies (10).

Case 3: $v_0 \notin \{\min(V), \max(V)\}$. Let e be the critical edge of W. By definition, e is of type 12.

Case 3a: The critical edge is a chord v_1v_2. Without loss of generality we may assume $\mathrm{val}_T(v_1) = 1$ and $\mathrm{val}_T(v_2) = 2$. Hence $v_1 > v_0 > v_2$. Now $v_0v_2 > v_1v_2$, so v_0v_2 is not of type 12. Hence v_0v_2 is of type 11 and $v_0v_2 \in T$. Then $v_2v_3 \in \overline{T}$, and $d(3) = 1$ (otherwise v_2v_3 is of type 22 and $v_2v_3 \in T$, which is not the case). Then v_2v_3 is of type 12. Since v_1v_2 is the critical edge, we have $v_1v_2 > v_2v_3$ and $v_1 < v_3$. In particular $v_1 \neq \max(V)$. Let $v_j = \max(V)$, and let P be the path from v_1 to v_j in T. If $v_0v_j \in T$, then P is the path (v_1, v_2, v_0, v_j), which satisfies (10). Otherwise, P is the path

$$(v_1,\, v_2,\, v_0,\, v_i,\, v_{i-1},\, \dots,\, v_j)$$

for some i. Therefore $\mathrm{val}_T(v_k) = 2$ for $i \leq k \leq j-1$, which implies that v_j and v_1 are respectively the largest and second largest vertices of P. Moreover, the chord $v_{j+1}v_j$ is of type 12, and v_1v_2 is the critical edge. So $v_{j+1} > v_2$ and P satisfies (10).

Case 3b: The critical edge is a radius v_0v_1. Without loss of generality we may assume $\mathrm{val}_T(v_2) = 2$ and $\mathrm{val}_T(v_k) = 1$. Hence $v_k > v_0 > v_2$ and $v_0v_1 < v_1v_2$. Thus v_1v_2 is not of type 12 and $\mathrm{val}_T(v_1) = 2$. Let j be the smallest number in $[1, k]$ such that $\mathrm{val}_T(v_j) = 1$, and let P be the path $(v_0, v_1, \dots, v_j)$. All the edges of P other than v_0v_1 and $v_{j-1}v_j$ are chords of type 22, thus belong to T. So $v_0v_k \in \overline{T}$ for $2 \leq k \leq j-1$, and $v_{j-1}v_j \in T$ (because $\mathrm{val}_T(v_{j-1}) = 2$). Thus $P \subset T$. Additionally, $j \geq 3$, and the endpoints of P, namely v_j and v_0, are respectively its largest and second largest vertex. Finally, $v_{j-1} > v_1$ (because v_0v_1, not v_0v_{j-1}, is the critical edge). Thus P satisfies (10). $\square$

Clearly, the only path on vertices $[1,4]$ satisfying (10) is 4213. There are two minimal paths on $[1,5]$, namely 53214 and 52314. (Others, such as 53124, contain subpaths satisfying (10), in this case 3124.) If we let $b(n)$ denote the number of minimal paths on vertices $[1,n]$ satisfying (10), then one can check that the sequence $b(4), b(5), \dots, b(9)$ is $1, 2, 5, 16, 61, 272$. This is a subsequence of the *Euler numbers* or *up/down numbers*, sequence A000111 in [11].

Many of the results of this section (such as Proposition 4.1) hold for all variable orderings which are "compatible" with the vertices of K_n, in the sense that if $v, v', v'' \in [1,n]$ and $v' < v''$, then $m_{vv'} > m_{vv''}$. One might also ask about other natural term orderings on R_n, such as *reverse lexicographic order*. This is given by $\prod m_{ij}^{a_{ij}} \underset{\text{rlex}}{>} \prod m_{ij}^{b_{ij}}$ if either $\sum a_{ij} > \sum b_{ij}$, or else $\sum a_{ij} = \sum b_{ij}$ and $a_{k\ell} < b_{k\ell}$, where $m_{k\ell}$ is the *smallest* variable for which $a_{k\ell} \neq b_{k\ell}$ (compare (9)). Then one can show that the leading trees of wheels with respect to reverse lex order are precisely those containing a path $v_1, \dots, v_k$, $k \geq 4$, such that $\max(v_1, \dots, v_k) = v_1$, $\max(v_2, \dots, v_k) = v_k$, and $v_2 < v_{k-1}$ (cf. Theorem 4.3).

5. The Stanley-Reisner Complex

We begin by recalling the definition of a simplicial complex and some related terminology; for more detail, see for instance [3, ch. 5]. Let V be a finite set of vertices. An (abstract) *simplicial complex on* V is a set Δ of subsets of V which contains as members all singleton sets $\{v\}$, $v \in V$, and with the property that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. The elements of Δ are called *faces*. A maximal face is called a *facet*. The *dimension* of a face F is $\dim F = |F| - 1$, and the dimension of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. A simplicial complex is a *simplex* if it has exactly one facet, and is *pure* if all its facets have the same cardinality.

Let $R = \mathbf{k}[x_1, \dots, x_n]$ and let $J \subset R$ be an ideal generated by squarefree monomials. Let $\Delta = \Delta(J)$ be the set of squarefree monomials which do not belong to J. Associating each squarefree monomial $x_{j_1} \dots x_{j_r}$ with the set $\{j_1, \dots, j_r\}$, we may regard Δ as a simplicial complex on vertices $[n]$, the *Stanley-Reisner complex* of J.

The Krull dimension of R/J is $1 + \dim \Delta$, and the degree of R/J (that is, its multiplicity as an R-module) equals the number of facets of Δ. More particularly, the Hilbert series of R/J corresponds to the *h-vector* of Δ, a combinatorial invariant about which we shall have more to say in Section 9.

We shall study the monomial ideal $J = J_n$ generated by the squarefree monomials corresponding to leading trees of wheels: that is,

$$J_n := \big(m_{LT(W)} \;\mid\; W \subset K_n \text{ a wheel}\big) \subset R_n. \tag{11}$$

Thus the Stanley-Reisner simplicial complex $\Delta(n)$ of J_n is defined by

$$\Delta(n) := \Delta(J_n) = \{E \subset E(K_n) \;\mid\; m_E \notin J_n\}. \tag{12}$$

Slightly more generally, if $V = \{v_1 < \cdots < v_n\}$ is a finite totally ordered set (typically, $V \subset \mathbb{N}$), we define a simplicial complex $\Delta(V)$ on $E(K_V)$, isomorphic to $\Delta(n)$, by replacing the edge ij with $v_i v_j$.

Note that the vertices of $\Delta(n)$ (in the sense of the definition of a simplicial complex) are the edges of K_n. By Theorem 4.3, an edge set E is a face of $\Delta(n)$ if and only if E contains no path of the form (10). For this reason, we will say that a path satisfying (10) is *forbidden*.

Example 5.1. No path with fewer than three edges satisfies (10), so the complexes $\Delta(2)$ and $\Delta(3)$ are simplices on $E(K_2)$ and $E(K_3)$ respectively. There is a unique forbidden path on four vertices, namely $4213 = \{12, 13, 24\}$, so the facets of $\Delta(4)$ are the edge subsets of K_4 that omit one edge from the path 4213, namely, $\{13, 14, 23, 24, 34\}$, $\{12, 14, 23, 24, 34\}$, and $\{12, 13, 14, 23, 34\}$. The faces of $\Delta(5)$ are those edge sets containing none of the paths 4213, 5213, 5214, 5314, 5324, 53214, or 52314. (These are precisely the paths satisfying (10) which are minimal under inclusion; see the discussion following the proof of Theorem 4.3.)

The main result on the structure of facets of $\Delta(n)$ is as follows.

Theorem 5.2. *Let $n \geq 3$, and let $\Delta(n)$ be the Stanley-Reisner simplicial complex just described. Then:*

(i) *$\Delta(n)$ is pure of codimension $2n-4$. Moreover, every facet $F \in \Delta(n)$ is 2-connected, and may be written uniquely as a disjoint union*

$$F \;=\; F^1 \sqcup F^2 \sqcup \{1n\}$$

with the following properties: each F^i is a facet of the complex $\Delta(V(F^i))$, and

$$\begin{aligned} &V(F^1) \cup V(F^2) = [1, n], \\ &V(F^1) \cap V(F^2) = \{\max(V(F^1))\}, \\ &1 \in V(F^1), \\ &n \in V(F^2). \end{aligned} \tag{13}$$

(ii) *Let F^1, F^2 be facets of the complexes $\Delta(V(F^1))$ and $\Delta(V(F^2))$ respectively, satisfying the four conditions just given. Then the edge set $F^1 \sqcup F^2 \sqcup \{1n\}$ is a facet of $\Delta(n)$.*

We will need several facts about connectivity and 2-connectivity of graphs, which we summarize here; for details see Chapter 4 of [15]. Let $G = (V, E)$ be a connected graph, and denote by $G - v$ the graph obtained by deleting a vertex v. If $G - v$ is disconnected, then v is called a *cut-vertex* of G (or of E). The vertex v *separates* the vertices w and x if w and x lie in different connected components of $G - v$. Equivalently, v lies on every path between w and x. G is called 2-*connected* if it has no cut-vertex. (In particular, we consider K_2 to be 2-connected, which is not the usual convention.)

A *block* of G is a maximal 2-connected subgraph of G. Every edge of G belongs to exactly one block. A vertex belongs to more than one block if and only if it is a cut-vertex; in addition, two blocks share at most one vertex. In particular, if $G' = (V', E')$ is a block of G, $v \in V'$ is not a cut-vertex of G, and $w \in V \setminus V'$, then there is a unique vertex $x \in V'$ such that x is a cut-vertex of G separating v and w.

If v is a cut-vertex of a connected graph G, then a v-*lobe* of G is a maximal connected subgraph not having v as a cut-vertex. Note that this decomposition is less fine than the block decomposition of G: for instance, if G is a path $v_1, v_2, \ldots, v_r$, then each edge is a block, but there are only two lobes with respect to each cut-vertex.

Finally, we will need the following special case of Menger's Theorem [15, p. 167]. Let v, w be nonadjacent vertices of a graph G. Then exactly one of the following conditions is true: either G contains a cut-vertex separating v and w, or else v, w lie on a common cycle C in G.

We now start our investigation of the simplicial complexes $\Delta(n)$. As noted in Example 5.1, $\Delta(2)$ and $\Delta(3)$ are simplices; in particular, $\Delta(n)$ is pure of dimension $2n-4$ for $n \leq 3$. For $n > 3$, the following criterion for nonmembership in $\Delta(n)$ will be very useful.

Lemma 5.3. *Let C be a cycle on vertices $V \subset [1,n]$. If $\min(V)$ and $\max(V)$ are not adjacent in C, then C contains a forbidden path.*

Proof. We proceed by induction on $|V|$. Without loss of generality we may assume $V = [1,n]$. If $n = 3$, the statement is vacuously true. If $n = 4$, then the only possibility is $C = 1,3,4,2,1$, which contains the forbidden path $4,2,1,3$.

Now suppose that $n > 4$, and the vertices $1, n$ are not adjacent. If C contains the edge $\{1, n-1\}$, then it contains a forbidden path of the form $n-1, 1, \ldots, n$. Otherwise, label the vertices of C in order as $v_1, \ldots, v_n, v_1$, with $v_1 = n$, $v_i = n-1$, and $v_j = 1$. In particular, $1 < i+1 < j < n$.

Let $r = n-i+1$, so that $n > r \geq 4$. By induction, the r-cycle $v_i, v_{i+1}, \ldots, v_n, v_i$ contains a forbidden path P. If $n-1$ and v_n are not adjacent in P, then $P \subset C$ and we are done. Otherwise, $n-1$ is an endpoint of P, with unique neighbor v_n. Let P' be the path obtained from P by deleting the edge $\{n-1, v_n\}$ and replacing it with $\{n, v_n\}$. Then P' is also forbidden, and $P' \subset C$ as desired. □

Corollary 5.4. *Let F be a 2-connected face of $\Delta(n)$, and let $V = V(F)$. Then the edge $\{\min(V), \max(V)\}$ belongs to F.*

Proof. Since F is 2-connected, it contains a cycle C supported at both $\min(V)$ and $\max(V)$. By Lemma 5.3, these vertices are adjacent in C, hence in F. □

By Theorem 4.3, no forbidden path in K_n contains either of the edges $\{1,n\}$ or $\{n-1,n\}$. Therefore, both of these edges belong to every facet of $\Delta(n)$. We will frequently work with the face $\hat{F} = F \setminus \{\{1,n\}\}$. Note that $\hat{F}$ is a face of $\Delta(n)$ of cardinality $|F| - 1$.

Lemma 5.5. *Let $n \geq 3$, and let F be a facet of $\Delta(n)$. Then both F and $\hat{F}$ are connected. However, $\hat{F}$ is not 2-connected; in particular, $\hat{F}$ has a cut-vertex separating 1 and n.*

Proof. Suppose that F is disconnected. Let e be an edge whose endpoints are the largest vertices of the connected components of F to which they belong. Then it is easy to check that $F + e \in \Delta(n)$, which contradicts the hypothesis that F is a facet.

Suppose that $\hat{F}$ is disconnected. Let H be the connected component of $\hat{F}$ containing the edge $\{n-1, n\}$. Then the vertex 1 does not belong to H; in particular, $e = \{1, n-1\} \notin F$. Let $F' = \hat{F} + e$. Either F' contains a forbidden path, or it is a face of $\Delta(n)$. If F' contains a forbidden path P, then P contains e, hence must be of the form $n-1, 1, \ldots, v, n$. But then $P - e \subset F$, which is impossible. On the other hand, if $F' \in \Delta(n)$, then $F' + \{1,n\} = F + e \in \Delta(n)$, which contradicts the hypothesis that F is a facet. We conclude that $\hat{F}$ is connected. However, the vertices 1 and n are not adjacent in $\hat{F}$, so by Lemma 5.3 they cannot lie in a common cycle. The last assertion of this lemma now follows from the special case of Menger's Theorem mentioned above. □

Lemma 5.6. *Let $n \geq 3$, and let F be a facet of $\Delta(n)$. Then the vertex 1 is not a cut-vertex of $\hat{F} = F - \{1, n\}$.*

Proof. We prove the contrapositive. Suppose that 1 is a cut-vertex of $\hat{F}$. Let A be the lobe containing vertex n, and let $B = F \setminus A$. Then neither A nor B is empty, and their only common vertex is 1. Let

$$x = \max(V(B)),$$
$$y = \min\{v \in V(A) \mid 1v \in \hat{F}\}.$$

Since $x \notin V(A)$, we have $x \neq y$. We consider the cases $x < y$ and $x > y$ separately. In both cases, the method of proof is to construct a forbidden path in $\hat{F}$, which is a contradiction because $\hat{F} \in \Delta(n)$.

Case 1: $x < y$. Suppose that $\hat{F} + xy$ contains a forbidden path P. Then $xy \in P$, since $\hat{F} \in \Delta(n)$ and $P \not\subset \hat{F}$. Label the vertices of P in order as $a, \ldots, x, y, \ldots, b$. Then $a, b \geq y > x$, so $a, b \in V(A)$. Hence the subpath of P from a to x must go through the vertex 1. Therefore P decomposes as $P_1 \sqcup P_2 \sqcup \{xy\} \sqcup P_3$, where P_1 is a path from a to 1, P_2 is a path from 1 to x, and P_3 is a path from y to b. But then $P_1 \sqcup \{1y\} \sqcup P_3$ is a forbidden path in $\hat{F}$. It follows that no such path P exists, and $\hat{F} + xy \in \Delta(n)$ But then so $F + xy$ is also an element of $\Delta(n)$, which contradicts the hypothesis that F is a *facet* of $\Delta(n)$.

Case 2: $x > y$. The face $\hat{F}$ contains a path y, 1, $x_1, \ldots, x_r = x$; call this P. If $x_1 < y$, then the subpath of P from y to x_i is forbidden, where i is the smallest index such that $x_i > y$; this is impossible. Therefore $x_1 > y$. On the other hand, A contains a path P' of the form

$$y = y_1,\, y_2,\, \ldots,\, y_s = n$$

with $y_i \neq 1$ for all i (because 1 is not a cut-vertex of A). Let t be the smallest number such that $y_t > x_1$. Then $\hat{F}$ contains the forbidden path

$$x_1,\, 1,\, y = y_1,\, y_2,\, \ldots,\, y_t,$$

which is a contradiction. □

By this lemma and the remarks following the statement of Theorem 5.2, each facet F has a unique vertex $a = a(F)$ with the following properties: a is a cut-vertex of $\hat{F}$ separating 1 and n, and a and 1 belong to the same 2-connected block of $\hat{F}$.

Lemma 5.7. *Let F be a facet of $\Delta(n)$, and let $a = a(F)$ be the vertex just described. Let F^1 be the a-lobe of $\hat{F}$ containing 1, and let $F^2 = F \setminus F^1$ be the union of all other a-lobes. Then $a = \max(V(F^1))$.*

Proof. We first prove a weaker statement. Let H be the unique block of F containing 1 (and thus a); we will show that $a = \max(V(H))$. Suppose not, and let $m = \max(V(H)) > a$. Then H contains the edge $1m$ by Corollary 5.4. Since H is connected, it contains a path P_1 from 1 to a which does not go through m. Moreover, F^2 contains a path from a to n, which we can truncate at the first vertex $v > m$ to obtain a path P_2. Then $\{1m\} \sqcup P_1 \sqcup P_2$ is a forbidden path in F from m to v. This is a contradiction, so $a = m$.

We now prove the full lemma. Suppose that $m = \max(V(F^1)) > a$. By the weaker case, $m \notin V(H)$, so there is a unique cut-vertex $b \in V(H)$ separating 1 and m. Then $b \neq a$ (otherwise $m = n \in V(F^1)$, which contradicts the definition of F^1) and $b \neq 1$ (by Lemma 5.6). Therefore $1 < b < a < m$. Then $1a \in H$ by Corollary 5.4. Moreover, H contains a path P from 1 to b which does not go through a. Additionally, $F^1 \setminus H$ contains a path from b to m, which we truncate at the first vertex greater than a to obtain a path P'. Then F contains a forbidden path, namely $\{1a\} \sqcup P \sqcup P'$; this is a contradiction. □

With all of these technical results in hand, we now proceed to the proof of the main theorem characterizing facets of the Stanley-Reisner complex $\Delta(n)$.

Proof of Theorem 5.2. Let a, F^1 and F^2 be as defined in Lemma 5.7. To complete the proof of the theorem, we must show that each F^i is 2-connected and is a facet of the complex $\Delta(V(F^i))$, and that the decomposition is unique. This last assertion is equivalent to the statement that a is the only cut-vertex of $\hat{F}$.

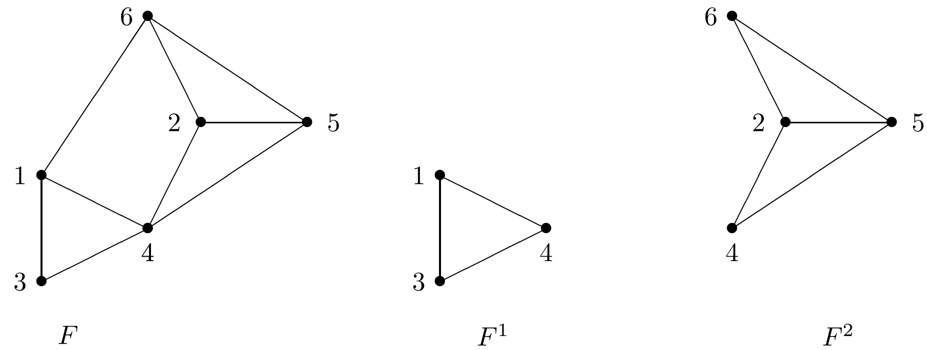
Suppose that F^1 is not a facet of $\Delta(V(F^1))$. That is, there is some edge $e \notin F^1$ such that $F^1 \cup \{e\} \in \Delta(V(F^1))$. Note that $e \notin F$. Since F is a facet of $\Delta(n)$, the edge set $F \cup \{e\}$ must contain a forbidden path P. Indeed, $P \subset \hat{F} \cup \{e\}$, since no forbidden path contains the edge $\{1, n\}$. On the other hand, $P \not\subset F^1 \cup \{e\}$ and $P \not\subset F^2$, since both $F^1 \cup \{e\}$ and F^2 are faces of $\Delta(n)$. Since both these edge sets are connected, it follows that P must go through a and have an endpoint $b \in V(F^1) \setminus \{a\}$. But by Lemma 5.7, the endpoints of P are not its two greatest vertices. Therefore P is not forbidden, and no such e exists. It follows that F^1 is a facet of $\Delta(V(F^1))$, as desired. The same argument shows that F^2 must be a facet of $\Delta(V(F^2))$.

We now show by induction on n that F is 2-connected and has cardinality $2n-3$, and that a is the only cut-vertex of $\hat{F}$. The base case $n = 3$ is easy: the only facet of $\Delta(3)$ is $F = \{12, 13, 23\}$. This has the right cardinality and is 2-connected, and $F - 13$ has a unique cut-vertex, namely 2. Now suppose $n > 3$. By induction F^1 and F^2 are 2-connected, and they share the vertex a, so a is the only cut-vertex of $\hat{F}$. Furthermore, F^1 and F^2 are precisely the 2-connected components of $\hat{F}$. Since $F = \hat{F} + 1n$, and this edge has one endpoint in each of $V(F^1) \setminus \{a\}$ and $V(F^2) \setminus \{a\}$, it follows that F is 2-connected. Also, $|V(F^2)| = n - |V(F^1)| - 1$, so by induction

$$|F| \;=\; 1 + \left(2\left|V(F^1)\right| - 3\right) + \left(2n - 2\left|V(F^1)\right| - 1\right) \;=\; 2n - 3. \tag{14}$$

Finally, suppose that F^1 and F^2 are facets of $\Delta(V(F^1))$ and $\Delta(V(F^2))$, respectively, and that the vertex sets $V(F^i)$ satisfy the conditions (13). So $a = \max(V(F^1))$ is a cut-vertex of $F^1 \cup F^2$. Suppose that F contains a forbidden path P. Note that $1n \notin P$ and neither F^1 nor F^2 contains P as a subset, so P must have an endpoint in $V(F^1) \setminus \{a\}$, which contradicts the condition $a = \max(V(F^1))$. Therefore $F \in \Delta(n)$. Moreover, $|F| = 2n - 3$ by (14), so F is a facet. □

Example 5.8. The edge set F shown below is a facet of the complex $\Delta(6)$; this can be checked routinely using the characterization of forbidden paths in Theorem 4.3. The subsets F^1 and F^2 are shown as well. Note that 4 is the unique cut-vertex of $\hat{F} = F - 16$, and that the corresponding lobes are exactly F^1 and F^2.



6. Representing Facets as Binary Trees

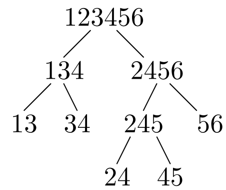
Let $F \in \Delta(n)$ be a facet. Iterating the decomposition of Theorem 5.2, we can construct a binary tree corresponding to F. In this section, we characterize these trees explicitly. Where no confusion can arise, we abbreviate sets of one-digit integers by a single word, e.g., $13467 = \{1, 3, 4, 6, 7\}$.

We begin by reviewing some general facts about planar trees. A *rooted tree* is a tree $T = (V, E)$ with a distinguished vertex $\mathrm{rt}(T)$, the *root* of T. We refer to the vertices of a rooted tree as *nodes*; this is in order to avoid confusion when the nodes are labeled with sets of vertices of K_n. We sometimes abuse notation by writing $v \in T$ instead of $v \in V$. If v, w are distinct nodes of T, we say that v is an *ancestor* of w (equivalently, w is a *descendant* of v) if v lies on the unique path from $\mathrm{rt}(T)$ to w. If v is an ancestor of w and $vw \in E$, then v is the *parent* of w and w is a *child* of v. In this case we write $v = v^{(P)}$. Two vertices with the same parent are called *siblings*. We denote by $T|v$ the subtree of T consisting of the node v and all its descendants.

A *rooted planar tree* is a rooted tree in which, for every node $v \in V$, the set of children of v is equipped with a total ordering, the *birth ordering*. That is, we can say which of two siblings (or two subtrees whose roots are siblings) is *older* than the other. We denote the ith oldest child of v by $v^{(i)}$. This notation can be iterated: for instance, $v^{(22)}$ means the second child of the second child of v. A node w is said to be *firstborn* if it is the oldest child of its parent; that is, $w = w^{(P1)}$. If $v = \mathrm{rt}(T)^{(j_1 \cdots j_s)}$, then we call the sequence of numbers (j_i) the *pedigree* of v. A *binary tree* is a rooted planar tree in which each node has either zero or two children. The *traversal* $\mathrm{trav}(T)$ of a rooted planar tree T is the list of nodes of T in the following order: first $\mathrm{rt}(T)$, then the nodes of $T^{(1)}$ in traversal order, then the nodes of $T^{(2)}$ in traversal order, and so on. (This is the order in which the nodes would be visited in the course of a depth-first search.)

Definition 6.1. Let $F \in \Delta(V)$ be a facet. Define a binary tree $\mathbf{T}(F)$ recursively as follows. If $|V| = 2$, then $\mathbf{T}(F)$ has one node, labeled by V itself. Otherwise, $\mathbf{T}(F)$ is the binary tree with root V, older subtree $\mathbf{T}(F^1)$, and younger subtree $\mathbf{T}(F^2)$, where F^1, F^2 are as in Theorem 5.2. $\mathbf{T}(F)$ is called the *decomposition tree* of F.

Example 6.2. Let F be the facet shown in Example 5.8. The decomposition tree of F has root node 123456, and by our computation of F^1 and F^2, the children of the root are 134 and 2456. Continuing in this way, we may calculate the complete decomposition tree, which is as follows:



There is a bijection $\mathbf{E}$ from nodes X of $\mathbf{T}(F)$ to edges of F, given by

$$\mathbf{E}(X) = \{\min(X), \max(X)\}.$$

We also write $\mathbf{E}(T) = \{\mathbf{E}(X) \mid X \in T\}$, where T is a tree or a set of nodes. The construction of $\mathbf{T}(F)$ and Theorem 5.2 immediately yield the following facts about decomposition trees.

Proposition 6.3. *Let $\mathbf{T}(F)$ be the decomposition tree of a facet $F \in \Delta(V)$. Then each node is labeled with at least two vertices of V (for short, $|X| \geq 2$), and the leaves of $\mathbf{T}(F)$ are exactly those nodes X for which $|X| = 2$. Each node X that is not a leaf satisfies the following conditions:*

(a) $\min(X) \in X^{(1)} \setminus X^{(2)}$;
(b) $\max(X) \in X^{(2)} \setminus X^{(1)}$;
(c) $X^{(1)} \cup X^{(2)} = X$;
(d) $X^{(1)} \cap X^{(2)} = \{\max(X^{(1)})\}$.

A tree satisfying these conditions will be called *admissible*.

Theorem 6.4. *Let $V \subset \mathbb{N}$, $|V| \geq 2$. Let $\mathrm{Adm}(V)$ be the set of all admissible binary trees with root V. Then the function $\mathbf{T}$ is a bijection from facets of $\Delta(V)$ to $\mathrm{Adm}(V)$. Moreover, the functions $\mathbf{T}$ and $\mathbf{E}$ are inverses.*

Proof. Note that $\mathbf{E}(T) \in \Delta(V)$ for all $T \in \mathrm{Adm}(V)$, by Theorem 5.2. Next, we show that the function $\mathbf{E}$ is injective on $\mathrm{Adm}(V)$. Let $T \neq T'$ be admissible trees with the same root V. Then there are nodes X, X' of T, T' respectively which have the same pedigree but different labels. Choose these two nodes as close to the root as possible, so that their parents have the same label Y. Then the edge sets $E - \mathbf{E}(Y)$ and $E' - \mathbf{E}(Y)$ belong to different blocks, so the facets $E = \mathbf{E}(T|Y)$ and $E' = \mathbf{E}(T'|Y)$ are distinct and $\mathbf{E}(T) \neq \mathbf{E}(T')$ as desired. It is clear from the definitions that $\mathbf{E}(\mathbf{T}(F)) = F$ for every facet F, so we are done. □

We may now speak of a *decomposition tree on* V as shorthand for a *decomposition tree of facets of* $\Delta(V)$.

Remark 6.5. It follows from Proposition 6.3 that for each parent node X in a decomposition tree, one has $X^{(2)} = (X \setminus X^{(1)}) \cup \{\max(X^{(1)})\}$. In particular, each older child determines its younger sibling uniquely, a fact that will be useful later.

7. Binary Total Partitions

By Theorem 6.4, the degree of the ideal J_n is the number of decomposition trees with root V. However, if we try to calculate this number directly using the conditions of Proposition 6.3, we wind up with an awkward recursive formula with no apparent closed-form solution. Instead, we construct a bijection from $\mathrm{Adm}(V)$ to a more easily enumerated set, the *binary total partitions* of V. We begin by defining these trees and listing some salient properties.

Definition 7.1. Let $V \subset \mathbb{N}$. A *binary total partition* of V is a binary tree T with nodes labeled by nonempty subsets of V, such that

- $\mathrm{rt}(T) = V$;
- If X is a leaf, then $|X| = 1$; and
- If X has children, then it is their disjoint union, and $\max(X) \in X^{(2)}$.

It would be simpler to define binary total partitions by ignoring the distinction between the two children of a given parent. However, the bijection between decomposition trees and binary total partitions will be easier to describe if we adopt the convention that each parent and its younger child have a common maximum.

The set of all binary total partitions of V is denoted $\mathrm{BPP}(V)$. It is elementary to show that

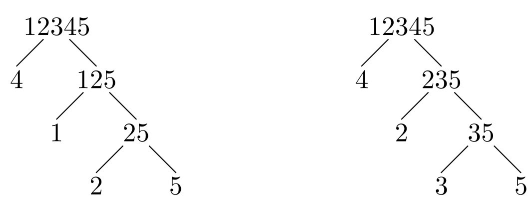
$$|\,\mathrm{BPP}(V)| \;=\; \frac{(2n-4)!}{2^{n-2}(n-2)!} \;=\; (2n-5)(2n-7)\cdots(3)(1). \tag{15}$$

where $n = |V| + 1$ [13, Example 5.2.6].

For $T \in \mathrm{BPP}(V)$ and $j \notin V$, define $\mathrm{Aug}_j(T)$ to be the tree obtained by adding the element j to the root of T (*augmenting T by j*). Clearly Aug_j is a bijection from $\mathrm{BPP}(V)$ to $\mathrm{BPP}(V, j) = \{\mathrm{Aug}_j(T) \;\mid\; T \in \mathrm{BPP}(V)\}$. We will say that $T \in \mathrm{BPP}(V, j)$ is *proper* if $j < \min(V)$.

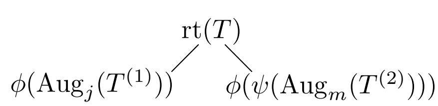
Let $S \subset \mathbb{N}$, $j \notin S$, $j' = \min(S \cup \{j\})$, and $S' = S \cup \{j\} \setminus \{j'\}$. Define a *straightening map* $\psi_{jj'}$ on trees $T \in \mathrm{BPP}(S, j)$ as follows. For each $k \in S$ with $k < j$, replace all occurrences of k in labels of non-root nodes of T with the next largest member of $S \cup \{j\}$. Note that $\psi_{jj'}$ is a bijection from $\mathrm{BPP}(S, j)$ to $\mathrm{BPP}(S', j')$; in particular, ψ_{jj} is the identity map. In what follows, it is frequently convenient and unambiguous to write simply ψ instead of $\psi_{jj'}$; it is only necessary to specify the subscripts if we want to work with the inverse function.

Example 7.2. Let $S = \{1, 2, 4, 5\}$ and $j = 3$. Put $j' = \min(S \cup \{j\}) = 1$ and $S' = S \cup \{j\} \setminus \{j'\} = \{2, 3, 4, 5\}$. The following figure shows a tree $T \in \mathrm{BPP}(S, j)$ and the tree $\psi_{jj'}(T) \in \mathrm{BPP}(S', j')$.



T $\qquad\qquad\qquad$ $\psi_{jj'}(T)$

We now construct a bijection between binary total partitions and decomposition trees. Let $S \subset \mathbb{N}$, $j < \min(S)$, and $T \in \mathrm{BPP}(S, j)$. Define a tree $\phi(T)$ recursively as follows. If $|S| = 1$, then $\phi(T) = T$. Otherwise, $\phi(T)$ is the tree



where $m = \max(\mathrm{rt}(T^{(1)}))$.

Theorem 7.3. *For every $S \subset \mathbb{N}$ and $j < \min(S)$, the function ϕ is a bijection from* $\mathrm{BPP}(S, j)$ *to* $\mathrm{Adm}(S \cup \{j\})$.

Proof. Let $T \in \mathrm{BPP}(S, j)$. Neither augmentation nor applying ψ changes the shape of a tree, so by induction $\phi(T)$ has the same shape as T. Accordingly, if X is a node of T, we may write $\phi(X)$ for the node of $\phi(T)$ in the same position as X.

First, we show that $\phi(T) \in \mathrm{Adm}(S \cup \{j\})$. If $|S| = 1$, this is trivial. Otherwise, we make the inductive assumption that whenever $|S'| < |S|$ and $j' < \min(S')$ (that is, the members of $\mathrm{BPP}(S', j')$ are proper) we have $\phi(T') \in \mathrm{Adm}(S' \cup \{j'\})$. In particular, the trees $\mathrm{Aug}_j(T^{(1)})$ and $\psi(\mathrm{Aug}_m(T^{(2)}))$ are proper, so by induction

$$\begin{array}{rclcl} \phi(T)^{(1)} & = & \phi(\mathrm{Aug}_j(T^{(1)})) & \in & \mathrm{Adm}(X^{(1)} \cup \{j\}) \quad \text{and} \\ \phi(T)^{(2)} & = & \phi(\psi(\mathrm{Aug}_m(T^{(2)}))) & \in & \mathrm{Adm}(X^{(2)} \cup \{m\}). \end{array}$$

It is now routine to check that $X = S \cup \{j\}$, $X^{(1)}$ and $X^{(2)}$ satisfy the conditions of Proposition 6.3.

Second, we show that ϕ is injective. If $|S| = 1$ or 2 then $|\,\mathrm{BPP}(S, j)| = 1$, so there is nothing to prove. Otherwise, we assume inductively that ϕ maps onto $\mathrm{BPP}(S', j)$ whenever $|S'| < |S|$. Let $T, U \in \mathrm{BPP}(S, j)$ with $\phi(T) = \phi(U)$. Then

$$\phi(\mathrm{Aug}_j(T^{(1)})) \;=\; \phi(T)^{(1)} \;=\; \phi(U)^{(1)} \;=\; \phi(\mathrm{Aug}_j(U^{(1)})).$$

By induction, we can cancel the ϕ's from the outer terms, and Aug_j is a bijection, so $T^{(1)} = U^{(1)}$. Now, setting $m = \max(\mathrm{rt}(T^{(1)})) = \max(\mathrm{rt}(U^{(1)}))$, we have

$$\phi(\psi(\mathrm{Aug}_m(T^{(1)}))) \;=\; \phi(T)^{(2)} \;=\; \phi(U)^{(2)} \;=\; \phi(\psi(\mathrm{Aug}_m(U^{(1)}))).$$

As before, it follows from the inductive hypothesis and the bijectivity of ψ and Aug_m that $T^{(2)} = U^{(2)}$. Of course $\mathrm{rt}(T) = \mathrm{rt}(U) = S \cup \{j\}$, so $T = U$ as desired.

Finally, we show that ϕ is surjective. If $|S| = 1$ or 2, then $|\,\mathrm{Adm}(S \cup \{j\})| = 1$, so there is nothing to prove. Otherwise, we make the inductive assumption that ϕ is surjective on $\mathrm{BPP}(S', j)$ whenever $|S'| < |S|$. Suppose that $j < \min(S)$ and $U \in \mathrm{Adm}(S \cup \{j\})$. Note that $U^{(1)} \in \mathrm{Adm}(S^{(1)} \cup \{j\})$ and $U^{(2)} \in \mathrm{Adm}(S^{(2)} \cup \{m\})$, where $S^{(1)}$ and $S^{(2)}$ are nonempty disjoint sets with $S^{(1)} \cup S^{(2)} = S$, $m = \max(S^{(1)})$, and $\max(S) \in S^{(2)}$. Let $k = \min(S^{(2)} \cup \{m\})$ and

$$\begin{array}{rclcl} T^{(1)} & = & \mathrm{Aug}_j^{-1}(\phi^{-1}(U^{(1)})) & \in & \mathrm{BPP}(S^{(1)}), \\ T^{(2)} & = & \mathrm{Aug}_m^{-1}(\psi_{mk}^{-1}(\phi^{-1}(U^{(2)}))) & \in & \mathrm{BPP}(S^{(2)}). \end{array}$$

Let T be the tree

$$\begin{array}{ccc} & S \cup \{j\} & \\ \diagup & & \diagdown \\ T^{(1)} & & T^{(2)} \end{array} \quad .$$

Then $\phi(T) = U$ and we are done. □

For convenience, we summarize here our present knowledge about the simplicial complex $\Delta(n)$. Note that $\dim I_n = \dim J_n = 2n - 3$ and $\mathrm{in}(I_n) \supset J_n$, so $\deg I_n = \deg \mathrm{in}(I_n) \leq \deg J_n$. Therefore:

Theorem 7.4. *The function $\theta = \phi \circ \mathrm{Aug}_1$ is a bijection from* $\mathrm{BPP}([2, n])$ *to* $\mathrm{Adm}([1, n])$*. In particular, $\Delta(n)$ has exactly*

$$\frac{(2n-4)!}{2^{n-2}(n-2)!}$$

facets. Moreover, this number equals the degree of the ideal J_n, and is an upper bound for the degree of I_n.

8. A Shelling of $\Delta(n)$

We prove in this section that the simplicial complex $\Delta(n)$ is shellable, hence Cohen-Macaulay, for all $n \geq 2$. We begin by recalling briefly the definition of shellability. (There are many equivalent definitions; see, e.g., [3, pp. 214–219].)

Definition 8.1. Let Δ be a pure simplicial complex. A total ordering $\succ$ of the facets of Δ is called a *shelling* if each facet F has a subset $\mathrm{sh}(F)$ such that
(SH1) If $F \succ G$, then $\mathrm{sh}(F) \not\subset G$, and
(SH2) For each $e \in \mathrm{sh}(F)$, there is some $G \prec F$ such that $F \setminus G = \{e\}$.

We first fix some notation. Let T be a decomposition tree. Denote by $\mathcal{L}(T)$ the set consisting of the root of T together with all firstborn nodes. For F a facet of $\Delta(n)$, we set $\mathcal{L}(F) = \mathcal{L}(\mathbf{T}(F))$.

Order the finite subsets of $\mathbb{N}$ as follows: $X > Y$ if $|X| > |Y|$, or if $|X| = |Y|$ and $\min(X \# Y) \in Y$. Let $F, G \in \Delta(n)$ be facets, with traversals

$$\begin{aligned} \mathrm{trav}(\mathbf{T}(F)) &= (X_1, \ldots, X_N), \\ \mathrm{trav}(\mathbf{T}(G)) &= (Y_1, \ldots, Y_N), \end{aligned}$$

where $N = 2n - 3$. Let $k = \min\{i \mid X_i \neq Y_i\}$. Then we define a total order on facets by putting $F \succ G$ if $X_k > Y_k$. Note that in this case X_k and Y_k have the same pedigree.

We will prove that this ordering satisfies the conditions **(SH1)** and **(SH2)**. First, we prove a fact to be used in the proof.

Lemma 8.2. *Let T be a decomposition tree on $V \subset \mathbb{N}$. Then the edge set $\mathbf{E}(\mathcal{L}(T))$ is a spanning tree of the complete graph on V.*

Proof. Suppose that $\mathbf{E}(\mathcal{L}(T))$ contains a cycle C. Let Y be the least common ancestor of the nodes in T corresponding to edges in C, and let $e = \mathbf{E}(Y)$. Then the endpoints of e are the minimum and maximum vertices of C, so e lies on C by Lemma 5.3. Let e' be the other edge of C incident to the vertex $\max(V(C)) = \max(Y)$. The node X corresponding to e' belongs to the *right rib* of $T|Y$, that is, the set

$$\mathrm{rr}(T) = \left\{ \mathrm{rt}(T),\ \mathrm{rt}(T)^{(2)},\ \mathrm{rt}(T)^{(22)},\ \ldots \right\}. \tag{16}$$

In particular $X \notin \mathcal{L}(T)$, which is a contradiction. Therefore $\mathbf{E}(\mathcal{L}(T))$ does not contain any cycles. Since its cardinality is $n-1$, it is a spanning tree of V. □

Theorem 8.3. *The order $\succ$ is a shelling order on the facets of $\Delta(V)$, with $\mathrm{sh}(F)$ defined recursively by*

$$\mathrm{sh}(F) = \begin{cases} \emptyset & \text{if } |V| \leq 3 \\ \mathrm{sh}(F^2) & \text{if } V(F^1) = \{\min(V), \min_2(V)\} \\ \mathrm{sh}(F^2) \cup \mathcal{L}(F^1) & \text{if otherwise} \end{cases}$$

where $\min_2(V)$ denotes the second smallest element of V.

Proof. We may assume without loss of generality that $V = [1, n]$. In the first part of the proof, we will show that the order $\succ$ satisfies **SH1**. Let F, G be facets of $\Delta(n)$ such that $F \succ G$. Put $T = \mathbf{T}(F)$, $U = \mathbf{T}(G)$, $\operatorname{trav}(T) = (X_1, \ldots, X_N)$, and $\operatorname{trav}(U) = (Y_1, \ldots, Y_N)$, with $X_i = Y_i$ for $i < k$ and $X_k > Y_k$. For ease of use, we abbreviate $X = X_k$ and $Y = Y_k$.

The parent nodes $X^{(P)} = Y^{(P)}$ have the same label, a set of cardinality ≥ 4. If X and Y are the younger siblings of their parents, then $X^{(P1)} = Y^{(P1)}$ by definition of k, but then $X = Y$ by Remark 6.5, a contradiction. Therefore $X = X^{(P1)}$ and $Y = Y^{(P1)}$. In particular $E \subset \operatorname{sh}(F)$, where $E = \mathbf{E}(\mathcal{L}(T|X))$. Now $X > Y$, so in particular $X \neq \{\min(X^{(P)}), \min_2(X^{(P)})\}$. Therefore $\mathbf{E}(X) \in E$.

We claim that $E \not\subset G$. It is sufficient to prove that $E \not\subset \mathbf{E}(U|Y^{(P)})$, because if $Z \in U$ is not a descendant of $Y^{(P)}$, then the endpoints of $\mathbf{E}(Z)$ do not both belong to $Y^{(P)} = X^{(P)}$, so $\mathbf{E}(Z) \notin E$.

Suppose first that $\max(X) > \max(Y)$. Then $\max(X) \notin Z$ for all $Z \in U|Y$; $\min(X) \notin Z$ for all $Z \in U|Y^{(P2)}$; and $\max(X) < \max(Y^{(P)})$. Hence $\mathbf{E}(X) \notin \mathbf{E}(U|Y^{(P)})$, establishing the claim.

Now, suppose that $\max(X) \leq \max(Y)$. Let $v \in X \setminus Y$. By Lemma 8.2, E is a spanning tree of the vertices in X. In particular, E contains a path P from v to $\min(X) = \min(Y)$. If $\max(X) = \max(Y)$, then $\max(Y)$ is a leaf of E (since $\mathbf{E}(X)$ is the only edge having it as an endpoint), while if $\max(X) < \max(Y)$, then $\max(Y) \notin V(E)$. In either case $\max(Y) \notin V(P)$. Note that $v \in Y^{(P2)} \setminus Y$ and $\min(Y) \in Y \setminus Y^{(P2)}$. By Theorem 5.2, $\max(Y)$ is a cut-vertex of $\mathbf{E}(U|Y^{(P)}) \setminus \{\mathbf{E}(Y^{(P)})\}$. So every path from $\min(Y)$ to v in $\mathbf{E}(U|Y^{(P)})$ must either pass through $\max(Y)$ or include the edge $\mathbf{E}(Y^{(P)})$. But P does neither of these things, so $P \not\subset \mathbf{E}(U|Y^{(P)})$, establishing the claim and completing the proof that $\succ$ satisfies **SH1**.

For the second part of the proof, let $F \in \Delta(n)$ be a facet, $T = \mathbf{T}(F)$, $e \in \operatorname{sh}(F)$, and X the node of T corresponding to e. We will show that $\Delta(n)$ contains a facet G satisfying condition **SH2**. We proceed by induction on n; there is nothing to prove if $n \leq 3$.

Suppose first that $e \in \operatorname{sh}(F^2)$. By induction, the simplicial complex $\Delta(V(F^2))$ has a facet $G^2 \prec F^2$ such that $F^2 \setminus G^2 = \{e\}$. Therefore, the facet of $\Delta(n)$ given by $F^1 \cup G^2 \sqcup \big\{\{\min(V(F)), \max(V(F))\}\big\}$ satisfies **SH2**.

On the other hand, suppose that $V(F^1) \neq \{1, 2\}$ and $e \in \mathcal{L}(F^1)$. We consider two separate cases: either X has children or it does not.

Case 1: X has children. Let $Y = X^{(P)}$, $E = \mathbf{E}(T|Y)$, $F_1 = \mathbf{E}(T|X^{(1)})$, and $F_2 = \mathbf{E}(T|X^{(2)}) \cup \mathbf{E}(T|Y^{(2)})$. Note that F_1 is a facet of $\Delta(X^{(1)})$, and $F_2 \in \Delta(X^{(2)} \cup Y^{(2)})$. Moreover, since $X^{(2)} \cap Y^{(2)} = \{\max(X^{(2)})\}$, we have

$$|F_2| \;=\; (2|X^{(2)}| - 3) + (2|Y^{(2)}| - 3) \;=\; \dim(\Delta(X^{(2)} \cup Y^{(2)}))$$

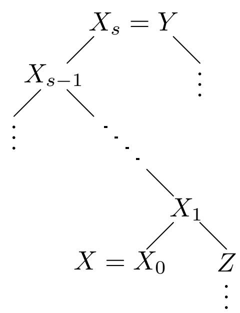
so $\Delta(X^{(2)} \cup Y^{(2)})$ has a facet F_2' of the form $F_2 \cup \{e'\}$. Let $G' = F_1 \cup F_2' \cup \{\mathbf{E}(Y)\}$; this is a facet of $\Delta(Y)$ because

$$X^{(1)} \cup (X^{(2)} \cup Y^{(2)}) = Y \quad \text{and} \quad X^{(1)} \cap (X^{(2)} \cup Y^{(2)}) = \{\max(X^{(1)})\}.$$

Let G be the facet of $\Delta(V)$ obtained by replacing $T|Y$ with $\mathbf{T}(G')$. Then $F \setminus G = \{e\}$. Moreover, the traversals of $\mathbf{T}(F)$ and $\mathbf{T}(G)$ first differ at $Y^{(1)}$, which is X in $\mathbf{T}(F)$ and $X^{(1)}$ in $\mathbf{T}(G)$. Since $X^{(1)} \subsetneq X$, we have $F \succ G$, so G satisfies **SH2**.

Case 2: X is a leaf. In particular $|X| = 2$. If $X \neq \{\min(X^{(P)}), \min_2(X^{(P)})\}$, then replacing X by that set produces a facet G which is easily seen to satisfy **SH2**.

Now suppose that $X = \{\min(X^{(P)}), \min_2(X^{(P)})\}$. Put $X_0 = X$ and $X_i = X_{i-1}^{(P)}$ for $i \geq 1$. Notice that $T^{(P)}$ is not in the right rib of T; if it were, it would have to be the root of T (since X is itself in the left subtree of T), which would imply that X is the only node in the left subtree, hence not a member of $\mathrm{sh}(F)$. Accordingly, let Y be the youngest ancestor of X such that $X \in T|Y^{(1)}$; then $Y = X_s$ for some $s \geq 2$. (If no such Y exists, then $X^{(P)} \in \mathrm{rr}(T)$, but $X \in \mathcal{L}(T)$ is in the left subtree of $\mathrm{rt}(T)$, so that $X^{(P)} = \mathrm{rt}(T)$.) Also, let $Z = X^{(P2)}$. That is, $T|Y$ has the following form:



For $i \in [s-1]$, define $E_i = \mathbf{E}(T|X_i) \setminus \{e\}$. Each $\mathbf{E}(T|X_i)$ is a facet of $\Delta(X_i)$, hence 2-connected by Theorem 5.2, so every E_i is connected. On the other hand, we claim that E_i is not 2-connected. More specifically, we claim that for every i, the vertex $m = \max(X_{s-1})$ is a cut-vertex separating $\min(X)$ and $\max(X)$. If $i = 1$, the vertex $\min(X)$ is incident to exactly two edges in $\mathbf{E}(T|X_1)$, namely $\mathbf{E}(X) = e$ and $\mathbf{E}(X_1) = \{\min(X_1), \max(X_1)\} = \{\min(X), m\}$, so m is the only neighbor of $\min(X)$ in E_1.

Now suppose that $1 < i \leq s-1$. Note that both $\mathbf{E}(X_1) = \{\min(X), m\}$ and $\mathbf{E}(Z) = \{\max(X), \max(Z) = m\}$ belong to E_i. Suppose that E_i contains a path P from $\min(X)$ to $\max(X)$ which does not pass through m. Let e' be the edge incident to $\min(X)$ in P, and W the corresponding node of T. Then W is not one of the nodes X_j, since $m = \max(X_j)$ for all j. Also, $W \notin T|Z$, because $\min(X) \notin Z$. Therefore $W \in T|X_j^{(1)}$ for some $j \in [2, i]$. In particular $\min(X) \in X_j^{(1)} \cap X_{j-1}$, so $\min(X) = \max(X_j^{(1)})$ and $\min(X) \neq \min(V(P))$. But then $P \cup \{\mathbf{E}(X_1), \mathbf{E}(Z)\}$ is a cycle in E_i in which the smallest vertex, namely $\min(V(P))$, and the largest vertex, namely m, are not adjacent. This contradicts Lemma 5.3 and completes the proof of the claim.

Let F_1 be the block of E_{s-1} containing the vertex $\min(Y)$, and let $F_2 = (E_{s-1} \setminus F_1) \cup \mathbf{E}(T|Y^{(2)})$. Also let $V_i = V(F_i)$ for $i = 1, 2$. Note that

$$V_1 \cup V_2 = Y, \tag{17a}$$
$$V_1 \cap V_2 = \{m\}, \quad \text{and} \tag{17b}$$
$$V_1 \subsetneq X_{s-1}. \tag{17c}$$

For $i = 1$ or 2, let F_i' be a facet of $\Delta(V(F_i))$ containing F_i. Now

$$|F_1| + |F_2| = |F_1 \cup F_2| = |\mathbf{E}(T|Y) \setminus \{\mathbf{E}(X), \mathbf{E}(Y)\}| = 2|Y| - 5$$

and

$$|F_1'| + |F_2'| = (2|V_1| - 3) + (2|V_2| - 3) = 2|Y| - 4,$$

the last equality following from (17a) and (17b). Therefore $F_1' \cup F_2' = F_1 \cup F_2 \cup \{e'\}$, for some edge $e' \neq e$, and the face $G' = F_1' \cup F_2' \cup \{\mathbf{E}(Y)\}$ is a facet of $\Delta(Y)$ with $\mathbf{E}(T|Y) \setminus G' = \{e\}$. Let G be the facet of $\Delta(V)$ obtained by replacing $T|Y$ with $\mathbf{T}(G')$. Then $F \setminus G = \{e\}$ as well. Furthermore, (17c) implies that $F \succ G$. Therefore G is the desired facet satisfying **SH2**. □

9. The h-vector of $\Delta(n)$

Since $\Delta(n)$ is shellable, its h-vector

$$h(\Delta(n)) \;=\; (h(n,0),\ h(n,1),\ \dots)$$

has the following combinatorial interpretation [3, Corollary 5.1.14]: for any shelling order $\succ$, $h(n,k)$ is the number of facets $F \in \Delta(n)$ with $|\operatorname{sh}_{\succ}(F)| = k$. In this section, we prove that the numbers $h(n,k)$ have another, more elementary combinatorial interpretation: they enumerate perfect matchings by the number of *long pairs*. These numbers were first investigated by Kreweras and Poupard [8]; we discovered the connection using the On-Line Encyclopedia of Integer Sequences [11]. We begin by obtaining a recurrence for $h(n,k)$, using the description of facets and the shelling order of Theorem 8.3. We then show that the recurrence is equivalent to one enumerating matchings by the number of long pairs

Let $F \in \Delta(n)$ be a facet and $T = \mathbf{T}(F)$ the corresponding decomposition tree. Define $\operatorname{sh}(T) \;=\; \{X \in T \;\mid\; \mathbf{E}(X) \in \operatorname{sh}(F)\}$.

Let $\theta : \mathrm{BPP}\,[2,n] \to \mathrm{Adm}([1,n])$ be the bijection of Theorem 7.4. Recall that T and $\theta(T)$ have the same shape, so we may define $\operatorname{sh}(U)$ to be the set of nodes in the same positions as the nodes in $\operatorname{sh}(\theta^{-1}(U))$.

Recall the definition (16) of the right rib $\operatorname{rr}(T)$ of a binary tree. As before, denote the second smallest element of a set S by $\min_2(S)$.

Lemma 9.1. *Let $F \in \Delta(n)$ be a facet, $T = \mathbf{T}(F)$ the corresponding decomposition tree, and X a firstborn node of T. Then:*

(i) *$X \notin \operatorname{sh}(T)$ if and only if $X = \{\min(X^{(P)}), \min_2(X^{(P)})\}$ and $X^{(P)} \in \operatorname{rr}(T)$.*
(ii) *$\theta(X) \notin \operatorname{sh}(\theta(T))$ if and only if $\theta(X) = \{\min(\theta(X)^{(P)})\}$ and $\theta(X)^{(P)} \in \operatorname{rr}(\theta(T))$.*

Proof. (i) The first condition implies that $X \notin \operatorname{sh}(X^{(P)})$; the second condition implies that $\operatorname{sh}(X^{(P)}) = \operatorname{sh}(F) \cap \mathbf{E}(T|X^{(P)})$. Together, they imply that $X \notin \operatorname{sh}(F)$. On the other hand, suppose that either or both conditions fail. Let Y be the youngest ancestor of X which does not belong to $\operatorname{rr}(T)$. Then Y is firstborn, and $X \in \operatorname{sh}(T|Y^{(P)}) \subset \operatorname{sh}(F)$.

(ii) This is obtained by translating the conditions from Lemma 9.1(i) into conditions on $\theta(T)$, using the definition of θ. □

Let $T \in \mathrm{BPP}\,[2, n-1]$ and $X \in T$. Define a tree $\gamma(T,X) \in \mathrm{BPP}\,[1, n-1]$ as follows: replace $T|X$ with the tree

$$\begin{array}{c} X \cup \{1\} \\ \swarrow \qquad \searrow \\ 1 \qquad\qquad X \end{array}$$

and append 1 to the label of every ancestor of X. The map

$$\gamma :\ \{(T,X) \;\mid\; T \in \mathrm{BPP}\,[2,n-1],\ X \in T\} \;\to\; \mathrm{BPP}\,[1,n-1]$$

is clearly one-to-one, and these sets have the same cardinality, so γ is a bijection. By incrementing all numbers in the labels of all nodes of $\gamma(T, X)$, we obtain a bijection $\tilde{\gamma}$ which maps pairs (T, X) as above to BPP $[2, n]$. Moreover,

$$\operatorname{sh}(\tilde{\gamma}(T,X)) \;=\; \begin{cases} \operatorname{sh}(T) & \text{if } X = \operatorname{rt}(T), \\ \operatorname{sh}(T) \cup \{1, X\} & \text{if } X \text{ is firstborn and } X \notin \operatorname{sh}(T), \\ \operatorname{sh}(T) \cup \{1\} & \text{otherwise,} \end{cases} \tag{18}$$

where $\operatorname{sh}(T) = \operatorname{sh}(\theta(T))$ for a binary total partition T. Accordingly, we may formulate a recurrence for the numbers $h(n, k)$. Suppose that $U = \tilde{\gamma}(T, X) \in$ BPP $[2, n]$ (so $T \in$ BPP $[2, n-1]$ has $2n-5$ nodes, of which $n-3$ are firstborn) and $|\operatorname{sh}(U)| = k$. Then one of the following is true: either

- $|\operatorname{sh}(T)| = k$ and $X = \operatorname{rt}(T)$;
- $|\operatorname{sh}(T)| = k-1$ and X is either one of the $n-3$ secondborn nodes of T, or one of the $k-1$ members of $\operatorname{sh}(T)$; or
- $|\operatorname{sh}(T)| = k-2$ and X is one of the $(n-3)-(k-2)$ firstborn nodes of T which do not belong to $\operatorname{sh}(T)$.

Therefore $h(n, k)$ is defined by the recurrence

$$h(n,k) \;=\; h(n-1,k) + (n+k-4)\,h(n-1,k-1) + (n-k-1)\,h(n-1,k-2) \tag{19}$$

with base cases $h(n, k) = 0$ if $k < 0$ or $k > n-2$, $h(n, 0) = 1$ for $n \geq 2$.

Let $n \in \mathbb{N}$. A *matching* on $[1, 2n]$ is a partition of $[1, 2n]$ into n pairs. A pair $\{i, j\}$ is *short* if $|i-j| = 1$; otherwise it is *long*. Define

$$\begin{aligned} M(n) &= \{\text{matchings on } [1,2n]\}, \\ M(n,k) &= \{X \in M(n) \mid X \text{ has } k \text{ long pairs}\}, \\ m(n,k) &= |M(n,k)|. \end{aligned}$$

Kreweras and Poupard [8] gave recurrences and a closed formula for $m(n, k)$. We will use a slightly different argument to show that $m(n, k)$ is given by a recurrence equivalent to (19).

Note first that $m(n, 0) = 1$ for all n, because the only matching on $[1, 2n]$ with no long pairs is $\big\{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\big\}$.

Let $X \in M(n-1)$ and $2 \leq p \leq 2n$. We define a matching $X^p \in M(n)$ by *inserting* the pair $\{1, p\}$ into X as follows. First, we relabel X to obtain a matching on $[1, 2n] \setminus \{1, p\}$; this amounts to replacing each $x \in [1, 2n-2]$ by $x+1$ if $x < p-1$, or by $x+2$ if $x \geq p-1$. Having done this, we obtain X^p by adjoining the pair $\{1, p\}$.

The map sending (X, p) to X^p is a bijection from $M(n-1) \times [2, 2n] \to M(n)$, so by induction $|M(n)| = (2n)!/2^n n!$. Moreover, we can derive a recurrence for $m(n, k)$. Let $X \in M(n, k)$. If $p = 2$, then $X^p \in M(n+1, k)$. If $\{p-2, p-1\} \in X$ (which occurs for $n-k$ values of p), then $X^p \in M(n+1, k+2)$ (because $\{p-2, p-1\}$ becomes the long pair $\{p-1, p+1\}$ in X^p). For the other $n+k$ values of p, we have $X^p \in M(n+1, k+1)$. Accordingly, $m(n, k)$ is defined by the recurrence

$$m(n,k) \;=\; m(n-1,k) + (n+k-2)\,m(n-1,k-1) + (n-k+1)\,m(n-1,k-2) \tag{20}$$

with base cases $m(n, k) = 0$ if $k < 0$ or $k > n$, and $m(n, 0) = 1$ for all $n \geq 0$.

Theorem 9.2. *Let $n \geq 2$ and $0 \leq k \leq n-2$. Then $h(n, k) = m(n-2, k)$, the number of matchings on $[1, 2n-4]$ with k long pairs.*

Proof. When $k = 0$ or $n = 2$, equality is immediate from the base cases. Otherwise, assume that $h(n', k) = m(n' - 2, k)$ for $n' < n$. Then (19) and (20) give

$$\begin{aligned} & m(n-2,k) \\ & \quad = \; m(n-3,k) + (n+k-4)\, m(n-3,k-1) + (n-k-1)\, m(n-3,k-2) \\ & \quad = \; h(n-1,k) + (n+k-4)\, h(n-1,k-1) + (n-k-1)\, h(n-1,k-2) \\ & \quad = \; h(n,k). \end{aligned}$$

□

10. A Recursive Lower Bound for Degree

In this section, we return to geometry. To complete the proof of the main theorem, we must show that J_n is the initial ideal of an ideal defining the slope variety $\tilde{\mathcal{S}}(K_n)$ scheme-theoretically. The key ingredient is to bound the (geometric) degree of $\tilde{\mathcal{S}}(K_n)$ from below. We do this by studying a family of *flattened slope varieties* $\tilde{\mathcal{S}}(n,k)$ forming a filtration of $\tilde{\mathcal{S}}(K_n)$: that is,

$$\tilde{\mathcal{S}}(K_n) \; = \; \tilde{\mathcal{S}}(n,1) \supset \tilde{\mathcal{S}}(n,2) \supset \cdots \supset \tilde{\mathcal{S}}(n,n).$$

We obtain a recursive lower bound $e(n,k) \leq \deg \tilde{\mathcal{S}}(n,k)$. In Section 11, we interpret the numbers $e(n,k)$ combinatorially and show that they actually give the degree exactly.

We begin by recalling some standard facts about the geometric notion of degree. Let $X \subset \mathbb{A}^N$ (or $\mathbb{P}^N$) be an algebraic set of dimension d. The *degree* of X, denoted $\deg X$, is the number of intersection points of X with a generic affine linear subspace of codimension d. We extend this definition to locally closed sets X, Y by putting $\deg X = \deg \overline{X}$. Degree is multiplicative on products: $\deg(X \times Y) = (\deg X)(\deg Y)$. Moreover, if H is a hyperplane, then $\deg(X \cap H) \leq \deg X$. Finally, if X is the union of locally closed sets $C_1, \ldots, C_k$ (for instance, if these are the irreducible components of X), then the degree of X is the sum of the degrees of all those C_i whose dimension equals $\dim X$.

We will also need some general facts about graph varieties; for details, see [10]. An *(affine) picture* $\mathbf{P}$ of K_n consists of n points $\mathbf{P}(1), \ldots, \mathbf{P}(n)$ in the plane $\mathbb{A}^2$, and $\binom{n}{2}$ non-vertical lines $\mathbf{P}(1,2), \ldots, \mathbf{P}(n-1,n)$, subject to the conditions $\mathbf{P}(i), \mathbf{P}(j) \in \mathbf{P}(ij)$ for all i, j. The set of all pictures is the *affine picture variety* $\tilde{\mathcal{V}}(K_n)$. The affine slope variety $\tilde{\mathcal{S}}(K_n)$ is thus obtained by projecting $\tilde{\mathcal{V}}(K_n)$ on affine coordinates corresponding to the slopes of the lines $\mathbf{P}(ij)$. We write ϕ for the surjection $\tilde{\mathcal{V}}(K_n) \twoheadrightarrow \tilde{\mathcal{S}}(K_n)$, and refer to a point in $\tilde{\mathcal{S}}(K_n)$ as a *slope picture* $\mathbf{m} = (m_{ij})$.

A *partition* of a set B is a collection of sets $\mathcal{A} = \{A_i\}$ such that B is the disjoint union of the A_i. For each partition $\mathcal{A}$ of $[1,n]$, there is a corresponding *cellule* $\tilde{\mathcal{V}}_{\mathcal{A}}(K_n)$ defined by

$$\tilde{\mathcal{V}}_{\mathcal{A}}(K_n) \; = \; \left\{ \mathbf{P} \in \tilde{\mathcal{V}}(K_n) \;\middle|\; \begin{array}{l} \mathbf{P}(i) = \mathbf{P}(j) \text{ if and only if} \\ i,j \text{ belong to the same part of } \mathcal{A} \end{array} \right\}.$$

Thus $\tilde{\mathcal{V}}(K_n)$ is the disjoint union of the cellules $\tilde{\mathcal{V}}_{\mathcal{A}}(K_n)$ as $\mathcal{A}$ ranges over all partitions. Of particular importance are the *discrete cellule*, corresponding to the partition of $[1,n]$ into n singleton sets, and the *indiscrete cellule*, corresponding to the partition with only one part.

Lemma 10.1. *Let* $\mathbf{m} \in \tilde{\mathcal{S}}(K_n)$*. Then there exists a picture* $\mathbf{P} \in \phi^{-1}(\mathbf{m})$ *in which not all points* $\mathbf{P}(1), \ldots, \mathbf{P}(n)$ *are the same. Equivalently,* ϕ *remains surjective if the indiscrete cellule is deleted from its domain.*

Proof. The equations defining a picture of G may be written in matrix form as $MX = 0$, where M is a matrix whose entries are linear forms in the variables m_e (as in (4)) and X is the column vector $[x_i - x_1]_{2 \leq i \leq n}$. Every maximal minor of M corresponds to a subset of $E(K_n)$ of cardinality $2n-2$, which must contain at least one rigidity circuit. If $\mathbf{m} \in \tilde{\mathcal{S}}(G)$, then every maximal minor of $M(\mathbf{m})$ vanishes. Therefore $M(\mathbf{m})$ has a nonzero nullvector X, which gives the x-coordinates (up to translation) of a picture $\mathbf{P}$ not in the indiscrete cellule. □

For $1 \leq k \leq n$, define an algebraic subset of $\tilde{\mathcal{S}}(K_n)$ by

$$\tilde{\mathcal{S}}(n,k) = \{\mathbf{m} \in \tilde{\mathcal{S}}(K_n) \;\mid\; m_{ij} = 0 \text{ for } 1 \leq i < j \leq k\}. \tag{21}$$

Note that $\tilde{\mathcal{S}}(n,1) = \tilde{\mathcal{S}}(K_n)$.

Proposition 10.2. *Let* $n \geq 2$ *and* $1 \leq k \leq n$*. Then*

$$\dim \tilde{\mathcal{S}}(n,k) \;=\; \begin{cases} 0 & \text{if } n = k, \\ 2n-k-2 & \text{if } n > k. \end{cases}$$

Proof. If $n = k$, then $\tilde{\mathcal{S}}(n,k)$ is a point. If $k = 1$, then $\dim \tilde{\mathcal{S}}(n,k) = \dim \tilde{\mathcal{S}}(K_n) = 2n-3 = 2n-k-2$. These two observations complete the proof for $n = 2$. For $n > 2$, we assume inductively that the formula holds for all smaller n and all k. For each partition $\mathcal{A}$ of $[1,n]$, define a closed subset $F_{\mathcal{A}}$ of $\tilde{\mathcal{V}}_{\mathcal{A}}(K_n)$ by

$$F_{\mathcal{A}} = \phi^{-1}(\tilde{\mathcal{S}}(n,k)) \cap \tilde{\mathcal{V}}_{\mathcal{A}}(K_n).$$

Note that Lemma 10.1 implies that $\tilde{\mathcal{S}}(n,k) = \bigcup_{\mathcal{A}} \phi(F_{\mathcal{A}})$, where $\mathcal{A}$ ranges over all partitions with at least two parts. In general, we may write $\mathcal{A}$ in the form

$$\mathcal{A} \;=\; \{A_1 \sqcup B_1,\; \ldots,\; A_s \sqcup B_s\}$$

where the A_i (resp. B_i) form a partition of $[1,k]$ (resp. $[k+1,n]$), with some A_i (resp. B_i) allowed to be empty. Let $a_i = |A_i|$ and $b_i = |B_i|$. We may order the parts of $\mathcal{A}$ in such a way that

$$\begin{array}{llll} a_i > 0, & b_i > 0 & \text{for} & i \in [1,p]; \\ a_i > 0, & b_i = 0 & \text{for} & i \in [p+1,q]; \\ a_i = 0, & b_i = 1 & \text{for} & i \in [q+1,r]; \\ a_i = 0, & b_i > 1 & \text{for} & i \in [r+1,s]. \end{array}$$

for some $p \leq q \leq r \leq s$. Note also that $q \leq k$. A picture $\mathbf{P} \in F_{\mathcal{A}}$ is therefore given by s distinct points in $\mathbb{A}^2$, of which q lie on a common horizontal line, together with the slopes of the lines $\mathbf{P}(e)$ such that the endpoints of the edge e are in the same part of $\mathcal{A}$ and are not both $\leq k$. This amounts to specifying s x-coordinates and $s-q+1$ y-coordinates for the points $\mathbf{P}(i)$; a point in $\tilde{\mathcal{S}}(a_i+b_i, a_i)$ for each $i \in [1,p]$; and a point in $\tilde{\mathcal{S}}(K_{b_i})$ for each $i \in [r+1,s]$. By the inductive hypothesis, we have

$$\begin{aligned}
\dim\, F_{\mathcal{A}} &= (2s-q+1)+\sum_{i=1}^{p}(2b_i-a_i-2)+\sum_{i=r+1}^{s}(2b_i-3)\\
&= -2p-s-q+3r+1+2\left(\sum_{i=1}^{p}b_i+\sum_{i=r+1}^{s}b_i\right)-\sum_{i=1}^{p}a_i\\
&= 3r-2p-q-s+1+2(n-k-(r-q))-\sum_{i=1}^{p}a_i\\
&= 2n-2k-2p+q+r-s+1-\sum_{i=1}^{p}a_i\\
&\leq 2n-2k-2p+q+r-s+1\\
&= (2n-k+1)+(-k-2p+q+r-s)\\
&\leq 2n-k+1
\end{aligned}$$

because $r \leq s$ and $q \leq k$. Every fiber of ϕ has dimension at least 3, so $\dim \phi(F_{\mathcal{A}}) \leq 2n-k-2$. Since $\tilde{\mathcal{S}}(n,k)$ is the disjoint union of the $F_{\mathcal{A}}$, its dimension is at most $2n-k-2$ as well. On the other hand, if $\mathcal{A}$ is the discrete partition into n singleton sets, then $p=0$, $q=k$, and $r=s=n$. So equality holds throughout the preceding calculation, and $\dim \tilde{\mathcal{S}}(n,k) = 2n-k-2$. □

Define

$$\tilde{\mathcal{S}}'(n,k) = \left\{\mathbf{m} \in \tilde{\mathcal{S}}(n,k) \;\mid\; m_{1,k+1}=0\right\}.$$

This set is the intersection of $\tilde{\mathcal{S}}(n,k)$ with a hyperplane, so its degree is a lower bound for that of $\tilde{\mathcal{S}}(n,k)$. We will calculate the degree of $\tilde{\mathcal{S}}'(n,k)$ by identifying its irreducible components of maximal dimension. First, we calculate the dimension of $\tilde{\mathcal{S}}'(n,k)$.

Proposition 10.3. *Suppose $n > k \geq 1$ and $n \geq 2$. Then $\dim \tilde{\mathcal{S}}'(n,k) = 2n-k-3$.*

Proof. Intersecting $\tilde{\mathcal{S}}(n,k)$ with a codimension-1 hyperplane can lower the dimension by at most 1. Hence for all n and k, we have $\dim \tilde{\mathcal{S}}'(n,k) \geq 2n-k-3$.

For the reverse inequality, we induct on n. If $n=2$ and $k=1$, then $2n-k-3=0$, and indeed $\tilde{\mathcal{S}}'(2,1) = \tilde{\mathcal{S}}(2,2)$ is a point. Now suppose that $n>2$ and that the formula holds for all smaller n and all k. For each partition $\mathcal{A}$ of $[1,n]$, define a closed subset $F'_{\mathcal{A}}$ of $\tilde{\mathcal{V}}_{\mathcal{A}}(K_n)$ by

$$F'_{\mathcal{A}} = \phi^{-1}\left(\tilde{\mathcal{S}}'(n,k)\right) \cap \tilde{\mathcal{V}}_{\mathcal{A}}(K_n).$$

By Lemma 10.1, $\tilde{\mathcal{S}}'(n,k) = \bigcup_{\mathcal{A}} \phi(F'_{\mathcal{A}})$, where $\mathcal{A}$ ranges over all partitions of n with at least two parts. We need to show that $\dim F'_{\mathcal{A}} \leq 2n-k$ for all such $\mathcal{A}$. In general, we may write

$$\mathcal{A} \;=\; \{\{k+1\} \sqcup A_1 \sqcup B_1,\; A_2 \sqcup B_2,\; \ldots,\; A_s \sqcup B_s\}$$

where the A_i (resp. B_i) form a partition of $[1,k]$ (resp. $[k+2,n]$), with some A_i (resp. B_i) allowed to be empty. Let $a_i = |A_i|$ and $b_i = |B_i|$. Order the parts of $\mathcal{A}$

so that for some $p \le q \le r \le s$,

$$\begin{array}{llll} a_i > 0, & b_i > 0 & \text{for} & i \in [2,p]; \\ a_i > 0, & b_i = 0 & \text{for} & i \in [p+1,q]; \\ a_i = 0, & b_i = 1 & \text{for} & i \in [q+1,r]; \\ a_i = 0, & b_i > 1 & \text{for} & i \in [r+1,s]. \end{array}$$

If $1 \in A_1$, then a picture $\mathbf{P} \in F'_{\mathcal{A}}$ is given by the following data: s distinct points in $\mathbb{A}^2$, of which q lie on a common horizontal line (so there are $s-q+1$ x-coordinates and s y-coordinates); a point in $\tilde{\mathcal{S}}'(1+a_1+b_1, a_1)$; a point in $\tilde{\mathcal{S}}(a_i+b_i, a_i)$ for each $i \in [2,p]$; and a point in $\tilde{\mathcal{S}}(K_{b_i})$ for each $i \in [r+1,s]$. By induction,

$$\dim F'_{\mathcal{A}} \;=\; (2s-q+1) + (a_1+2b_1-1) + \sum_{i=2}^{p}(a_i+2b_i-2) + \sum_{i=r+1}^{s}(2b_i-3).$$

Now a simple calculation, which we omit because of its similarity to that of Proposition 10.2, shows that $\dim F'_{\mathcal{A}} \le 2n-k$ as desired.

If $1 \notin A_1$, then a picture $\mathbf{P} \in F'_{\mathcal{A}}$ is given by the following data: s distinct points in $\mathbb{A}^2$, of which q lie on a common horizontal line; a point in $\tilde{\mathcal{S}}(a_i+b_i, a_i)$ for each $i \in [1,p]$; and a point in $\tilde{\mathcal{S}}(K_{b_i})$ for each $i \in [r+1,s]$. Therefore

$$\begin{aligned} \dim F'_{\mathcal{A}} &= (2s-q+1) + \sum_{i=1}^{p}(a_i+2b_i-2) + \sum_{i=r+1}^{s}(2b_i-3) \\ &= (-2p-q+3r-s+1) + \sum_{i=1}^{p} a_i + 2\left(\sum_{i=1}^{p} b_i + \sum_{i=r+1}^{s} b_i\right) \\ &= 2n-2k-2p+q+r-s-1+\sum_{i=1}^{p} a_i. \end{aligned}$$

Note that $\sum_{i=1}^{p} a_i \le k-(q-p)$, since $\{A_1,\dots,A_q\}$ is a partition of $[1,k]$ with no empty parts. Therefore $\dim F'_{\mathcal{A}} \;\le\; 2n-k-p+r-s-1 \;<\; 2n-k$ as desired. □

Theorem 10.4. *Let $n \ge 2$ and $k \in [1,n]$. Then*

$$\deg \tilde{\mathcal{S}}(n,k) \ge e(n,k),$$

where $e(n,n)=1$ for all n; $e(2,1)=1$; and

$$\begin{aligned} e(n,k) \;&=\; e(n,k+1) \;+\; e(n-1,k-1) \\ &+\; \sum_{t=1}^{k-1}\sum_{u=0}^{n-k-2}\binom{k-1}{t}\binom{n-k-1}{u}\; e(t+u+1,t)\; e(n-t-u,k-t+1). \end{aligned}$$

Proof. For all n, the variety $\tilde{\mathcal{S}}(n,n)$ is a point, so it has degree $1 = e(n,n)$. The variety $\tilde{\mathcal{S}}(2,1) = \tilde{\mathcal{S}}(K_2)$ is a line, whose degree is again $1 = e(2,1)$. Now suppose that $n > k$ and $n > 2$. As noted before, $\deg \tilde{\mathcal{S}}'(n,k) \le \deg \tilde{\mathcal{S}}(n,k)$. The idea of the proof is to write $\tilde{\mathcal{S}}'(n,k)$ as a finite union of locally closed subsets. Once we have done this, we may obtain a lower bound for $\deg \tilde{\mathcal{S}}'(n,k)$ by summing lower bounds for the degrees of those locally closed subsets of dimension $2n-k-3$.

First, note that $\tilde{\mathcal{S}}'(n,k) \supset \tilde{\mathcal{S}}(n,k+1)$, and that this second set has dimension $2n-k-3$.

Second, let $\mathbf{m} = (m_{ij})$ be a slope picture in $\tilde{\mathcal{S}}'(n,k) \setminus \tilde{\mathcal{S}}(n,k+1)$. That is, $m_{ij} = 0$ for $1 \leq i < j \leq k$, and $m_{1,k+1} = 0$, but $m_{2,k+1}, \dots, m_{k,k+1}$ are not all zero. Define two sets of vertices $T(\mathbf{m})$, $U(\mathbf{m})$ by

$$\begin{aligned} T(\mathbf{m}) &= \{i \in [2,k] \ \mid \ m_{i,k+1} \neq 0\}, \\ U(\mathbf{m}) &= \left\{i \in [k+2,n] \ \mid \ \mathbf{P}(i) = \mathbf{P}(k+1) \text{ for all } \mathbf{P} \in \phi^{-1}(\mathbf{m})\right\}. \end{aligned}$$

Then $T(\mathbf{m}) \neq \emptyset$, and $\tilde{\mathcal{S}}'(n,k) \setminus \tilde{\mathcal{S}}(n,k+1)$ is the disjoint union of the sets

$$S_{T,U} = \left\{\mathbf{m} \in \tilde{\mathcal{S}}'(n,k) \setminus \tilde{\mathcal{S}}(n,k+1) \ \mid \ T(\mathbf{m}) = T, \ U(\mathbf{m}) = U\right\}$$

for $\emptyset \neq T \subset [2,k]$ and $U \subset [k+2,n]$. On $\phi^{-1}(S_{T,U})$, one has $\mathbf{P}(i) = \mathbf{P}(k+1)$ for all $i \in T \cup U$. Thus the data for a picture in $\phi^{-1}(S_{T,U})$ consists of $n - |T| - |U|$ points, of which $k - |T| + 1$ must lie on a common horizontal line, together with the slopes of lines corresponding to edges with both endpoints in $T \cup U$. Applying the surjective map ϕ, we see that $S_{T,U}$ has the form

$$S_{T,U} \cong \tilde{\mathcal{S}}(n-t-u, k-t+1) \times \tilde{\mathcal{S}}(t+u+1, t) \tag{22}$$

where $t = |T|$ and $u = |U|$. If $U \neq [k+2,n]$, then $n - t - u > k - t + 1$. Applying Proposition 10.2 to (22), one obtains

$$\begin{aligned} \dim\, S_{T,U} &= (2(n-t-u) - (k-t+1) - 2) + (2(t+u+1) - t - 2) \\ &= 2n - k - 3. \end{aligned}$$

If $U = [k+2,n]$, then $n - t - u = k - t + 1$. Now the first factor in (22) is a single point, so applying Proposition 10.2 yields

$$\dim\, S_{T,U} \ = \ 2(t+u+1) - t - 2 \ = \ (2n-k-3) + (t-k+1).$$

This is strictly less than $2n - k - 3$ unless $T = [2,k]$, in which case $S_{T,U} \cong \tilde{\mathcal{S}}(n-1, k-1)$. Putting all this together, we obtain

$$\begin{aligned} \deg\, \tilde{\mathcal{S}}'(n,k) \ \geq \ & \deg\, \tilde{\mathcal{S}}(n,k+1) + \deg\, \tilde{\mathcal{S}}(n-1,k-1) \\ & + \sum_{\substack{\emptyset \neq T \subset [2,k] \\ U \subsetneq [k+2,n]}} \deg\, \tilde{\mathcal{S}}(n-t-u, k-t+1) \ \deg\, \tilde{\mathcal{S}}(t+u+1, t). \end{aligned}$$

Now, summing over t and u instead of T and U, and multiplying the summand by the appropriate binomial coefficients to reflect the number of ways of choosing the sets T and U, one obtains the desired recurrence. □

11. Decreasing Planar Trees

The recurrence established in Theorem 10.4 has a combinatorial interpretation in terms of *decreasing planar trees*. We begin with some general facts about these objects, recalling the general definitions and terminology for rooted planar trees given in Section 6.

Definition 11.1. Let V be a finite subset of $\mathbb{N}$. A *decreasing planar tree on* V is a rooted planar tree T with the following property: if v, w are nodes of T with v an ancestor of w, then $v > w$. (In particular, T is rooted at the maximum element of V.) The set of all decreasing planar trees on V is denoted $\mathrm{DPT}(V)$; as usual we abbreviate $\mathrm{DPT}([1,n])$ by $\mathrm{DPT}(n)$.

The number of decreasing planar trees is given, once again, by the double factorial numbers:

$$|\operatorname{DPT}(n)| \;=\; (2n-3)(2n-5)\cdots(3)(1) \;=\; \frac{(2n-2)!}{(n-1)!\;2^{n-1}}.$$

(This is an elementary combinatorial exercise; see, e.g., [12, pp. 13–16].) We will obtain a recurrence enumerating decreasing planar trees on V by the *largest leaf* statistic: $L(T) = \max\{i \in V \mid i \text{ is a leaf}\}$. Let

$$\begin{aligned} \operatorname{DPT}(n,k) &= \{T \in \operatorname{DPT}(n) \mid L(T) \le k\}, \\ d(n,k) &= |\operatorname{DPT}(n,k)|. \end{aligned}$$

Note that $d(n,1) = 1$ for all n, since if $L(T) = 1$ then the tree T can only be the path in which each $j > 1$ has the unique child $j-1$. Moreover, if $n \ge 2$, then $\operatorname{DPT}(n,n) = \operatorname{DPT}(n,n-1) = \operatorname{DPT}(n)$. In addition, $d(n,k) = 0$ if $n > 0$ and $k \notin [1,n]$. Conventionally, we put $d(0,k)$ for all k.

If T_1 and T_2 are rooted planar trees and X is a node of T_2, we may graft T_1 to T_2 by attaching it as the oldest subtree of X. We denote the resulting tree by $T_1 \underset{X}{\nearrow} T_2$, suppressing the subscript if $X = \operatorname{rt}(T_2)$ (the most common case). To illustrate this operation, let T be the binary tree pictured in Example 6.2. Let T_1 be the subtree rooted at the node labeled 245, and let T_2 be the tree obtained from T by deleting T_1. Then $T = T_1 \underset{X}{\nearrow} T_2$, where X is the node labeled 2456.

Suppose that $T_1 \in \operatorname{DPT}(V_1)$ and $T_2 \in \operatorname{DPT}(V_2)$, where $V_1 \cap V_2 = \emptyset$. If $X \in V_2$ and $\operatorname{rt}(T_1) < X$, then $T_1 \underset{X}{\nearrow} T_2 \in \operatorname{DPT}(V_1 \cup V_2)$. In addition, every tree T with more than one node can be written uniquely as $T_1 \nearrow T_2$, by taking $T_1 = T|\operatorname{rt}(T)^{(1)}$ and $T_2 = T - T_1$.

Lemma 11.2. *Let $k \in [2,n]$. Then*

$$d(n,k-1) \;=\; \sum_{a=1}^{k-1} \binom{k-1}{a}\, d(a,a)\; d(n-a,k-a).$$

Proof. Let T_1 be a decreasing planar tree with nodes $A \subset [1,k-1]$, $|A| = a < k$. There are $\binom{k-1}{a} d(a,a)$ such trees. Meanwhile, let T_2 be a decreasing planar tree with nodes $[a+1,n]$ and largest leaf $L(T_2) \le k$. Subtracting a from the label of every node gives a bijection between such trees T_2 and the set $\operatorname{DPT}(n-a,k-a)$. Therefore the number of pairs (T_1,T_2) is counted by the right-hand side of the desired equality. For each such pair, let

$$f(T_1,T_2) = T_1 \underset{k}{\nearrow} T_2.$$

The map f is clearly injective, and $f(T_1,T_2) \in \operatorname{DPT}(n,k-1)$. On the other hand, for any $T \in \operatorname{DPT}(n,k-1)$, let $T' = T|k^{(1)}$ and $T'' = T - T'$. Then $T' \underset{k}{\nearrow} T'' = T$. So f maps onto $\operatorname{DPT}(n,k-1)$, and the desired equality follows. $\square$

This formula may be taken as a recursive definition for the numbers $d(n,k)$. For small values of n and k, they are given by the following table.

	$k=1$	2	3	4	5	6
$n=2$	1	1				
3	1	3	3			
4	1	7	15	15		
5	1	15	57	105	105	
6	1	31	105	561	945	945

Lemma 11.3. *Let* $1 < k \leq n$. *Then*

$$\sum_{a=1}^{k-1}\sum_{c=1}^{n-k-1}\binom{k-1}{a}\binom{n-k-1}{c}\ d(a+c,a)\ d(n-a-c,k-a)\ =$$
$$\sum_{w=0}^{k-2}\sum_{y=1}^{n-k-1}\binom{k-1}{w}\binom{n-k-1}{y}\ d(w+y,w+1)\ d(n-w-y,k-w-1).$$

Proof. First, we interpret the sums on either side of the desired equality as enumerating certain kinds of decreasing planar trees. The left-hand side counts trees of the form $T = T' \nearrow T'' \in \mathrm{DPT}(n)$ such that

$$\begin{aligned} V(T') &= A \sqcup C, & L(T') &\in A, \\ V(T'') &= B \sqcup D \sqcup \{k,n\}, & L(T'') &\in B \sqcup \{k\}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} A \sqcup B &= [1,k-1], & 1 \ \leq\ a = |A| \ \leq\ k-1, \\ C \sqcup D &= [k+1,n-1], & 1 \ \leq\ c = |C| \ \leq\ n-k-1. \end{aligned}$$

Meanwhile, the right-hand side counts trees of the form $U = U' \nearrow U'' \in \mathrm{DPT}(n)$ such that

$$\begin{aligned} V(U') &= W \sqcup Y, & L(T') &\in W \sqcup \{\min(Y)\}, \\ V(U'') &= X \sqcup Z \sqcup \{k,n\}, & L(T'') &\in X, \end{aligned} \tag{24}$$

where

$$\begin{aligned} W \sqcup X &= [1,k-1], & 0 \leq w = |W| \ \leq\ k-2, \\ Y \sqcup Z &= [k+1,n-1], & 1 \leq y = |Y| \ \leq\ n-k-1. \end{aligned}$$

Second, we show how to perform "surgery" on a tree of the form (23) to obtain a tree of the form (24). Suppose that $T = T' \nearrow T''$ satisfies (23). Define a tree $f(T)$ as follows.

- *If k has children in T*, then $f(T) = T$. In this case $B \neq \emptyset$, so $A \subsetneq [1,k-1]$. Also, $L(T) < k$; in particular $L(T'') \in B$. Therefore, T satisfies (24), with $W = A$, $X = B$, $Y = C$, $Z = D$.
- *If k is a leaf*, then we form $f(T)$ by detaching all subtrees of $\min(C)$ and reattaching them to k in the same birth order. Note that there is at least one such subtree since $\min(C) > L(T)$. Also, $L(f(T)) = \min(C)$, and $f(T)$ satisfies (24), with

$$\begin{aligned} W &= A \setminus \{\text{descendants of } \min(C)\}, & Y &= C, \\ X &= B \sqcup \{\text{descendants of } \min(C)\}, & Z &= D. \end{aligned}$$

Third, we show how this surgery can be reversed. Suppose that $U = U' \nearrow U''$ satisfies (24). Define a tree $g(U)$ as follows:

- *If* $\min(Y)$ *has children in* U, then $g(U) = U$. In this case $W \neq \emptyset$. Also, $L(T) < k$; in particular $L(T'') \in X$. Therefore, T satisfies (24), with $A = W$, $B = X$, $C = Y$, $D = Z$.
- *If* $\min(Y)$ *is a leaf*, then we form $g(U)$ by detaching all subtrees of k and reattaching them to $\min(Y)$ in the same birth order. Note that since $k > L(U)$, this step is not trivial. Also, $L(g(U)) = k$, and $g(U)$ satisfies (23), with

$$\begin{aligned} A &= W \sqcup \{\text{descendants of } \min(Y)\}, & C &= Y, \\ B &= X \setminus \{\text{descendants of } \min(Y)\}, & D &= Z. \end{aligned}$$

Finally, we show that the functions f and g are mutual inverses. Suppose that T satisfies (23). In particular, $\min(C)$ has children in T, so if $f(T) = T$, then $g(f(T)) = T$. On the other hand, if k is a leaf in T, then by construction $\min(C) = \min(Y)$ is a leaf in $f(T)$, and the two surgeries described above are inverses by definition. So $g(f(T)) = T$ in any case. The proof that $f(g(U)) = U$ is analogous. So f and g are bijections and we are done. $\square$

Lemma 11.4. *Let* $1 < k \leq n$. *Then*

$$d(n,k) = d(n-1,k) + \sum_{a=1}^{k-1} \sum_{c=0}^{n-k-1} \binom{k}{a}\binom{n-k-1}{c} d(a+c,a)\ d(n-a-c,k-a).$$

Proof. Deleting the node n gives a bijection between trees $T \in \mathrm{DPT}(n,k)$ such that n has exactly one child, and $\mathrm{DPT}(n-1,k)$. This accounts for the term $d(n-1,k)$ on the right-hand side. On the other hand, suppose that $T \in \mathrm{DPT}(n,k)$ is a tree in which n has more than one child. Then $T = T' \nearrow T''$, where

$$\begin{aligned} V(T') &= A \sqcup C, & L(T') &\in A, \\ V(T'') &= B \sqcup D \sqcup \{n\}, & L(T'') &\in B, \\ A \sqcup B &= [1,k], & 1 \leq a &= |A| \leq k-1, \\ C \sqcup D &= [k+1,n-1], & 0 \leq c &= |C| \leq n-k-1. \end{aligned}$$

Conversely, if a pair (T', T'') satisfies these conditions, then $T = T' \nearrow T''$ belongs to $\mathrm{DPT}(n,k)$, and n has at least two children in T. These pairs are enumerated by the double sum on the right-hand side of the desired equality. $\square$

Applying Pascal's identity

$$\binom{k}{a} = \binom{k-1}{a} + \binom{k-1}{a-1}$$

to the equation of Lemma 11.4, we obtain

$$\begin{aligned} d(n,k) \ &= \ d(n-1,k) \\ &+ \sum_{a=1}^{k-1} \sum_{c=0}^{n-k-1} \binom{k-1}{a}\binom{n-k-1}{c}\ d(a+c,a)\ d(n-a-c,k-a) \\ &+ \sum_{a=1}^{k-1} \sum_{c=0}^{n-k-1} \binom{k-1}{a-1}\binom{n-k-1}{c}\ d(a+c,a)\ d(n-a-c,k-a). \end{aligned} \tag{25}$$

Breaking off the $c = 0$ term of the first double sum in (25) yields

$$\begin{aligned} d(n,k) &= d(n-1,k) + \sum_{a=1}^{k-1} \binom{k-1}{a} d(a,a)\, d(n-a,k-a) \\ &+ \sum_{a=1}^{k-1} \sum_{c=1}^{n-k-1} \binom{k-1}{a} \binom{n-k-1}{c} d(a+c,a)\, d(n-a-c,k-a) \\ &+ \sum_{a=1}^{k-1} \sum_{c=0}^{n-k-1} \binom{k-1}{a-1} \binom{n-k-1}{c} d(a+c,a)\, d(n-a-c,k-a). \end{aligned} \tag{26}$$

Applying Lemma 11.2 to the single sum and Lemma 11.3 to the first double sum in (26), we obtain

$$\begin{aligned} d(n,k) &= d(n-1,k) + d(n,k-1) \\ &+ \sum_{w=0}^{k-2} \sum_{y=1}^{n-k-1} \binom{k-1}{w} \binom{n-k-1}{y} d(w+y,w+1)\, d(n-w-y,k-w-1) \\ &+ \sum_{a=1}^{k-1} \sum_{c=0}^{n-k-1} \binom{k-1}{a-1} \binom{n-k-1}{c} d(a+c,a)\, d(n-a-c,k-a). \end{aligned} \tag{27}$$

In the first double sum of (27), we may change the upper limit on y from $n-k-1$ to $n-k$, because the additional summand is zero; in addition put $x = n-k-y$. In the second double sum, put $w = a-1$ and $x = n-k-1-c$. We obtain

$$\begin{aligned} d(n,k) &= d(n-1,k) + d(n,k-1) \\ &+ \sum_{w=0}^{k-2} \sum_{x=0}^{n-k-1} \binom{k-1}{w} \binom{n-k-1}{x-1} d(n-k+w-x,w+1)\, d(k-w+x,k-w-1) \\ &+ \sum_{w=0}^{k-2} \sum_{x=0}^{n-k-1} \binom{k-1}{w} \binom{n-k-1}{x} d(n-k+w-x,w+1)\, d(k-w+x,k-w-1) \end{aligned} \tag{28}$$

Now, another application of Pascal's identity, together with the base cases mentioned previously, yields the following:

Proposition 11.5. *The numbers $d(n,k)$ satisfy the recurrence*

$$\begin{aligned} d(n,k) &= d(n-1,k) + d(n,k-1) \\ &+ \sum_{w=0}^{k-2} \sum_{x=0}^{n-k-1} \binom{k-1}{w} \binom{n-k}{x} d(n-k+w-x,w+1)\, d(k-w+x,k-w-1). \end{aligned}$$

Finally, we come to the geometric reason for all this enumeration. The recurrence given by Proposition 11.5 is easily turned into one giving the lower bounds for the degree of the flattened slope variety $\tilde{\mathcal{S}}(n,k)$, as we now show.

Theorem 11.6. *We have* $\deg \tilde{\mathcal{S}}(n,k) \geq d(n-1,n-k)$. *In particular,*

$$\deg \tilde{\mathcal{S}}(K_n) \geq d(n-1,n-1) = |\operatorname{DPT}(n-1)|.$$

Proof. Replacing $d(i,j)$ with $e(i+1,i-j+1)$ in Proposition 11.5, we obtain

$$\begin{aligned} e(n+1,n-k+1) \; &= \; e(n,n-k)+e(n+1,n-k+2) \\ &+ \sum_{w=0}^{k-2}\sum_{x=0}^{n-k-1}\binom{k-1}{w}\binom{n-k}{x} \\ &\quad \times \; e(n-k+w-x+1,n-k-x)\; e(k-w+x+1,x+2). \end{aligned}$$

Now, setting

$$n=n'-1, \qquad k=n'-k', \qquad x=k'-w'-1, \qquad w=x'$$

and removing the primes from the new variables, we obtain the recurrence of Theorem 10.4. It follows that $d(n,k)=e(n+1,n-k+1)$ for all n and k, or equivalently $e(n,k)=d(n-1,n-k)$. □

12. Proof of the Main Theorem

With all the pieces in place, we can now prove our main result.

Proof of Theorem 1.1. The variety $\tilde{\mathcal{S}}(K_n)$ is irreducible and reduced, hence is defined scheme-theoretically by the prime ideal $\sqrt{I_n}$. In addition, we have

$$J_n \subset \operatorname{in}(I_n) \subset \operatorname{in}(\sqrt{I_n}). \tag{29}$$

By Theorem 5.2 we have

$$\operatorname{codim} J_n = 2n-3 = \dim \tilde{\mathcal{S}}(K_n) = \operatorname{codim} \sqrt{I_n}$$

and by Theorem 7.4 and Theorem 11.6 we have

$$\deg J_n = \frac{(2n-4)!}{2^{n-2}(n-2)!} \leq \deg \sqrt{I_n}.$$

We have seen in Theorem 8.3 that the Stanley-Reisner complex of J_n is shellable. Hence J_n is Cohen-Macaulay [3, Theorem 5.1.13]. In particular it is unmixed, so any strictly larger ideal has either larger codimension or smaller degree. Meanwhile, $\operatorname{in}(\sqrt{I_n})$ has the same codimension and degree as $\operatorname{in}(\sqrt{I_n})$ by [5, Theorem 15.26], Therefore equality holds in (29). Since $J_n=\operatorname{in}(I_n)$ is Cohen-Macaulay, so is I_n [14, Corollary B.2.1]. It follows that $\tilde{\mathcal{S}}(K_n)$ is Cohen-Macaulay. Finally, the formula for the Hilbert series is a restatement of the results of Section 9. □

We remark that for the degree bounds in Theorem 11.6 to be sharp, equality must hold in the lower bounds of Theorem 10.4. That is, the flattened slope variety $\tilde{\mathcal{S}}(n,k)$ has degree exactly $d(n-1,n-k)$.

We conclude by mentioning some problems for further study. Using the computer algebra system *Macaulay* [1], we have verified that I_n is generated by the (cubic) tree polynomials of the $\binom{n}{4}$ copies of K_4 arising as subgraphs of K_n, for $n \leq 9$. Accordingly, we conjecture that they do so for all n. The difficulty is that there appears to be no canonical way to write a tree polynomial $\tau(W)$ as an R_{K_n}-linear combination of the $\tau(K_4)$'s, where W is a wheel with five or more vertices. Further computations using *Macaulay* suggest that the wheel polynomials may form a *universal Gröbner basis* for I_n (that is, a Gröbner basis with respect to *every* term ordering), but it is not clear how to prove this.

The technique of using the Stanley-Reisner complex to obtain information about graph varieties may be applicable to graphs other than K_n. The likeliest candidates are graphs with many symmetries, such as threshold graphs or complete bipartite graphs. The degree and Hilbert series of the slope variety of a graph are combinatorial invariants for which it would be nice to have a more elementary graph-theoretic description. In addition, it would be of interest to determine whether $\tilde{\mathcal{S}}(G)$ is always defined scheme-theoretically by the tree polynomials of rigidity circuit subgraphs of G, and whether it is always Cohen-Macaulay. (We have computed I_G for several small graphs; in all cases the Cohen-Macaulay property is satisfied.)

An anonymous referee has pointed out that the space $\mathscr{T}_n$ of *phylogenetic trees on n vertices*, constructed by Billera, Holmes and Vogtmann in [2], has degree $(2n-4)!/(2^{n-2}(n-2)!)$, the same as that of $\tilde{\mathcal{S}}(K_n)$. Moreover, for $0 < m < n$, the space $\mathscr{T}_n$ contains many subspaces isomorphic to $\mathscr{T}_m$ [2, p. 743], just as $\tilde{\mathcal{S}}(K_n)$ contains many copies of $\tilde{\mathcal{S}}(K_m)$ (see Section 10). It would be interesting to explore possible connections between pictures and phylogenetic trees.

Other spaces related to graph varieties include the Fulton-Macpherson *compactification of configuration space* [6] and the De Concini-Procesi *wonderful model of subspace arrangements* [4]. The results of this article might serve as a starting point for studying these relationships in more detail.

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