Math 821 Problem Set #7
Posted: Friday 4/29/11
Due date: Wednesday 5/11/11

Note: In all cases, "compute the homology groups" means "compute $H_n(X)$ for $n > 0$" – you don't have to incessantly repeat that $H_0(X) = \mathbb{Z}$ for path-connected spaces X.

Problem #1 [Hatcher p.156 #9b] Compute the homology groups of $S^1 \times (S^1 \vee S^1)$. (Note that this space is *not* homeomorphic to $(S^1 \times S^1) \vee (S^1 \times S^1)$.)

Problem #2 [Hatcher p.156 #9c] Compute the homology groups of the space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

Problem #3 [Hatcher p.156 #9d] Compute the homology groups of the quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Problem #4 [Hatcher p.155 #2, modified] Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Hint: Lift.)

Problem #5 [Hatcher p.156 #15] Show that if X is a CW complex then $H_n(X^n)$ is free, by identifying it with the kernel of the cellular boundary map $H^n(X^n, X^{n-1}) \to H^{n-1}(X^{n-1}, X^{n-2})$. (Hint: Once you understand how the diagram on p.139 is constructed, the proof is purely algebraic and should be quite short.)

Problem #6 [Hatcher p.158 #29] The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R.