Math 821 Problem Set #6
Posted: Friday 4/15/11
Due date: Wednesday 4/27/11

Problem #1 [Hatcher p.131 #4] Compute the simplicial homology groups of the "triangular parachute" obtained from the standard 2-simplex Δ^2 by identifying its three vertices to a single point.

Problem #2 [Hatcher p.131 #8] Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps.

First, arrange the tetrahedra in a cyclic pattern as in the figure (see p. 131) so that each T_i shares a common vertical face with its two neighbors. For consistent notation, call the top and bottom vertices x and y respectively, and call the side vertices $v_1, \ldots, v_n$, so the tetrahedra are

$$T_1 = [x, v_1, v_2, y],\ T_2 = [x, v_2, v_3, y],\ \ldots, T_{n-1} = [x, v_{n-1}, v_n, y],\ T_n = [x, v_n, v_1, y].$$

Second, identify the bottom face of T_i with the top face of T_{i+1} for all i, that is, $[v_i, v_{i+1}, y] = [v_{i+1}, v_{i+2}, x]$.

Show that the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}$, $\mathbb{Z}_n$ $(=\mathbb{Z}/n\mathbb{Z})$, 0, $\mathbb{Z}$ respectively. (Start by making a complete census of the oriented simplices, including a record of which ones have been identified — for example, $[x, v_1, v_2] = [y, v_n, v_1]$ is a triangle in X.)

Problem #3 [Hatcher p.131 #11] Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Problem #4 A [finite] *partially ordered set* or *poset* is a finite set P with an order relation $\leq$ such that for all $x, y, z \in P$: (1) $x \leq x$; (2) if $x \leq y$ and $y \leq x$, then $x = y$; and (3) if $x \leq y$ and $y \leq z$, then $x \leq z$. Of course, $x < z$ means that $z \leq z$ and $x \neq z$. If $x \leq z$ or $z \leq x$, we say that x, z are *comparable*. A *chain* in P (not to be confused with a simplicial or singular chain!) is a subset in which every two elements are comparable.

(#4a) Prove that the set $\Delta(P)$ of chains in P is a simplicial complex. (This is called the *order complex* of P.)

(#4b) Suppose that P has a unique maximal element. Prove that $\Delta(P)$ is contractible.

(#4c) For each $n \geq 1$, construct a poset for which $\Delta(P)$ is homeomorphic to an n-sphere.

(#4d) The *Möbius function* μ of P is defined as follows.

(1) Adjoin two new elements $\hat{0}, \hat{1}$ to P to obtain a poset $\hat{P}$, in which $\hat{0} < x < \hat{1}$ for every $x \in P$.
(2) Define μ recursively as follows: First, if x is minimal (i.e., there exists no y such that $x > y$) then $\mu(x) = -1$. Second, if $\mu(y)$ has already been defined for all $y < x$, then define

$$\mu(x) = -\sum_{y<x} \mu(y).$$

(So you can work out the values of μ on all elements of P by starting at the bottom and working your way up.)

Make a conjecture as to how the Euler characteristic of $\Delta(P)$ can be obtained from the Möbius function of P.

Some LaTeX tips

1. Matrices with borders

The `\bordermatrix` command can be used for matrices whose columns and rows you want to label. This can be useful for bookkeeping in a simplicial homology calculation. For example, the boundary map ∂_2 of the standard 3-simplex is

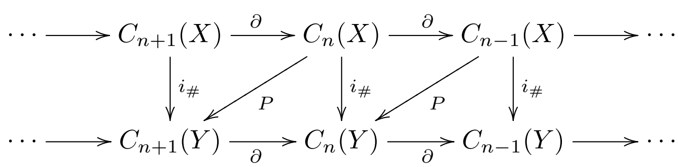
$$\bordermatrix{ & 123 & 124 & 134 & 234 \cr 12 & 1 & 1 & 0 & 0\cr 13 & -1 & 0 & 1 & 0\cr 14 & 0 & -1 & -1 & 0\cr 23 & 1 & 0 & 0 & 1\cr 24 & 0 & 1 & 0 & -1\cr 34 & 0 & 0 & 1 & 1}$$

which can be produced as follows:

```
$$\bordermatrix{
   & 123 & 124 & 134 & 234 \cr
12 &  1  &  1  &  0  &  0\cr
13 & -1  &  0  &  1  &  0\cr
14 &  0  & -1  & -1  &  0\cr
23 &  1  &  0  &  0  &  1\cr
24 &  0  &  1  &  0  & -1\cr
34 &  0  &  0  &  1  &  1}$$
```

2. Commutative diagrams

The `xypic` package provides a way to typeset commutative diagrams in LaTeX. For instance, consider the following diagram, which arises in the proof of Theorem 2.10 in Hatcher:



It can be typeset as follows:

```
$$\xymatrix{
\cdots\ar[r]
  & C_{n+1}(X) \ar[r]^{\bd} \ar[d]^{i_\#}
  & C_{n}(X)   \ar[r]^{\bd} \ar[d]^{i_\#} \ar[dl]^{P}
  & C_{n-1}(X) \ar[r]       \ar[d]^{i_\#} \ar[dl]^{P}
  &\cdots\\
\cdots\ar[r]
  & C_{n+1}(Y) \ar[r]_{\bd}
  & C_{n}(Y)   \ar[r]_{\bd}
  & C_{n-1}(Y) \ar[r]
  &\cdots}$$
```

This is like a `tabular` or `array` environment: the & symbols are delimiters between columns. The `\ar` commands create arrows emanating from the current cell in the table, with the code in [square brackets] specifying where the arrow should point; e.g., `\ar[dl]` makes an arrow pointing towards the cell one row down and one column left of the current cell.