Math 821 Problem Set #5
Posted: Friday 4/1/11
Due date: Wednesday 4/13/11

Problem #1 In class on Friday, I asserted that if G is a graph with n vertices and c connected components, and M is the signed vertex-edge incidence matrix of G, then $\operatorname{rank} M = n - c$. Prove this statement (over any ground field).

Problem #2 Fix a ground field $\mathbb{F}$ and a nonnegative integer n. Let V_k be the vector space with basis $\{\sigma_A\}$, where A ranges over all k-element subsets of $\{1, 2, \ldots, n\}$. Define a linear transformation $\partial_k : V_k \to V_{k-1}$ as follows: if $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$, then

$$\partial_k(\sigma_A) = \sum_{i=1}^{k} (-1)^{i+1} \sigma_{A \setminus \{a_i\}}.$$

(Having defined ∂_k on the basis elements, it extends uniquely to all of V_k by linearity.)

(#2a) Prove that $d_k \circ d_{k+1} = 0$ for all k. (Note: I know this calculation is done explicitly in Hatcher, but it is so important that everyone should do it for themselves at least once!) Conclude that

$$\operatorname{im} \partial_k \subseteq \ker \partial_{k+1}.$$

(#2b) For $n = 3$, write out the maps ∂_i as explicit matrices.

(#2c) Prove that for every k, the set $\{\partial_k(\sigma_A) : \ 1 \in A\}$ is a basis for the vector space $\operatorname{im} \partial_k$.

(#2d) Use (3) to prove that in fact $\operatorname{im} \partial_k = \ker \partial_{k+1}$. (Hint: By (1), all you have to show is that these vector spaces have the same dimension.)

Problem #3 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Describe $\operatorname{coker} A$ (i) if A is regarded as a linear transformation over $\mathbb{Q}$; (ii) if A is regarded as a linear transformation over $\mathbb{Z}$; (iii) if A is regarded as a linear transformation over $\mathbb{F}_q$ (the finite field with q elements).

Problem #4 Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over a field $\mathbb{F}$. A *squarefree monomial* in R is a product of distinct indeterminates (e.g., $x_1x_4x_5$, but not $x_1x_5^2$). Let I be an ideal generated by squarefree monomials of degree ≥ 2.

(#4a) Show that the set

$$\Delta = \{\sigma \subset [n] \mid \prod_{i \in \sigma} x_i \notin I\}$$

is an abstract simplicial complex on n vertices.

(This is called the *Stanley-Reisner complex* of I — or, alternately, I is the Stanley-Reisner ideal of Δ.)

(#4b) Describe Δ looks like in the case that I is (i) the zero ideal; (ii) generated by a single monomial of degree d; (iii) generated by all monomials of degree d for some $k \leq n$; (iv) (assuming $n = 2m$ is even) generated by the degree-2 monomials x_1x_2, x_3x_4, $\ldots$, $x_{2m-1}x_{2m}$.