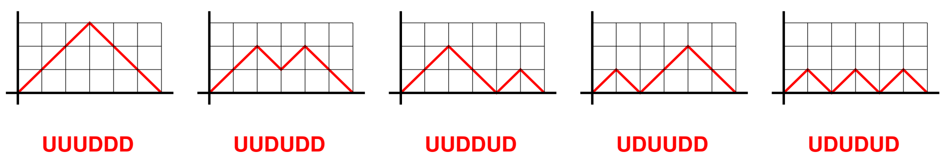
Counting Dyck Paths

A *Dyck path of length* $2n$ is a diagonal lattice path from $(0, 0)$ to $(2n, 0)$, consisting of n up-steps (along the vector $(1, 1)$) and n down-steps (along the vector $(1, -1)$), such that the path never goes below the x-axis. We can denote a Dyck path by a word $w_1 \dots w_{2n}$ consisting of n each of the letters D and U. The condition "the path never goes below the x-axis" is equivalent to "every initial subword $w_1 \dots w_k$ contains at least as many U's as D's."

For example, here are the five Dyck paths of length $2 \times 3 = 6$:



You have seen that the number of Dyck paths of length $2n$ is $\frac{1}{n+1}\binom{2n}{n}$. Here is a bijective proof.

First, take every Dyck path of length $2n$ and prepend[1] a U to it. In the world of lattice paths, what we now have is the set of lattice paths from $(-1, -1)$ to $(2n, 0)$ that begin with an up-step and never subsequently drop below the x-axis. We'll call these things *augmented Dyck paths.* Note that the number of augmented Dyck paths is the same as the number of Dyck paths, because you can just chop off the leading U.

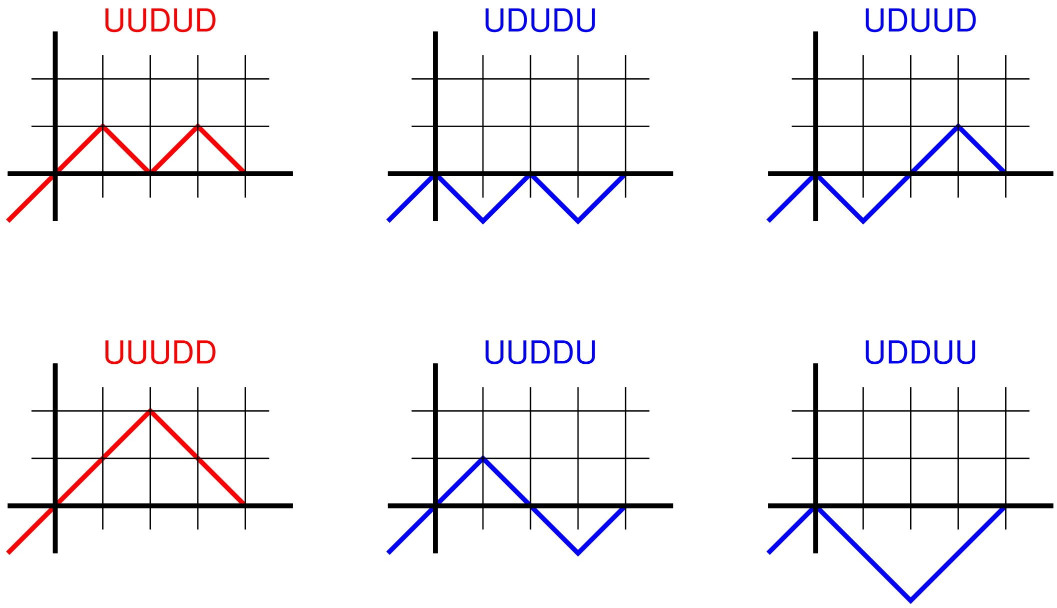
Now, let X_n denote the set of *all* lattice paths from $(-1, -1)$ to $(2n, 0)$ that begin with an up-step. The number of these is certainly $\binom{2n}{n}$.

If $w = w_1 \dots w_{2n+1} \in X$ and $w_k = U$, we say that the path $w_k \dots w_{2n+1} w_1 \dots w_{k-1}$ is *rotationally equivalent* to w. Rotational equivalence is an equivalence relation. For example, in X_2, the rotational equivalence classes are

$$\{UUDUD,\ UDUDU,\ UDUUD\}, \quad \{UUUDD,\ UUDDU,\ UDDUU\}.$$

Here is what X_2 looks like. Each row constitutes a rotational equivalence class. Only the red paths are augmented Dyck paths.

[1] I.e., stick it on the left. "Append" would mean to stick it on the right.



Theorem 0.1. *Every rotational equivalence class in X_n has exactly $n+1$ elements. Of these, exactly one is an augmented Dyck path. Therefore, there is a bijection between Dyck paths and rotational equivalence classes.*

Proof. First, every equivalence class has at most $n+1$ members, since each path in X contains $n+1$ up-steps.

Suppose that rotating by k places fixes a word $w \in X_n$. Consider the equivalence relation on $[2n+1]$ given by $i \sim j$ iff $i \equiv j + kx \pmod{2n+1}$ for some $x \in \mathbb{Z}$. For each equivalence class under $\sim$, all of the letters in those positions must be the same. On the other hand, the cardinality of each equivalence class is $m = (2n+1)/\gcd(2n+1, k)$ (check this). Therefore the number of U's and the number of D's must both be multiples of m. But these numbers are $n+1$ and n respectively, which are coprime. So $m = 1$ and k is a multiple of $2n+1$, but that implies that the only rotations that fix w are trivial. Therefore, all equivalence classes have cardinality $n+1$.

Let Q be the rightmost absolute minimum of w (i.e., of all the points on the path with minimum y-coordinate, choose the rightmost one). In particular, the step following this point is an up-step. Consider the rotationally equivalent path that starts at Q. It never goes below the x-axis — if it did, then there's either a lower point to the left of Q or a point at least as low to its left. On the other hand, Q is uniquely defined, so for any other possible starting point R, the point Q is either strictly lower, or equally low and further right, so the rotationally equivalent path starting at R is not an augmented Dyck path. □

Corollary 0.2. *The number of Dyck paths is $\frac{1}{n+1}\binom{2n}{n}$.*