Classical Greek mathematics through Pythagoras

The great achievement of the Greek mathematicians was developing the idea of proof. As opposed to their Babylonian and Egyptian predecessors who were mostly concerned with how to solve practical problems, the Greeks were interested in why mathematics worked.

The mathematicians in ancient Babylon and Egypt were priests and government officials[1] and focused on practical administrative problems: measurement of land, division of goods, tax assessment, architecture, etc. The intended audience of mathematical texts was probably other administrators, and so there was no perceived need to explain why the rules worked—just how to use the rules. The excerpt from the Rhind Papyrus quoted last week is an excellent example of this.

By contrast, many Greek mathematicians were independently wealthy and had spare time on their hands to concern themselves with knowledge for its own sake. On the other hand, the Greeks discovered that this abstract knowledge could often be put to practical use: e.g., measuring the height of the Great Pyramid, or estimating the circumference of the earth.

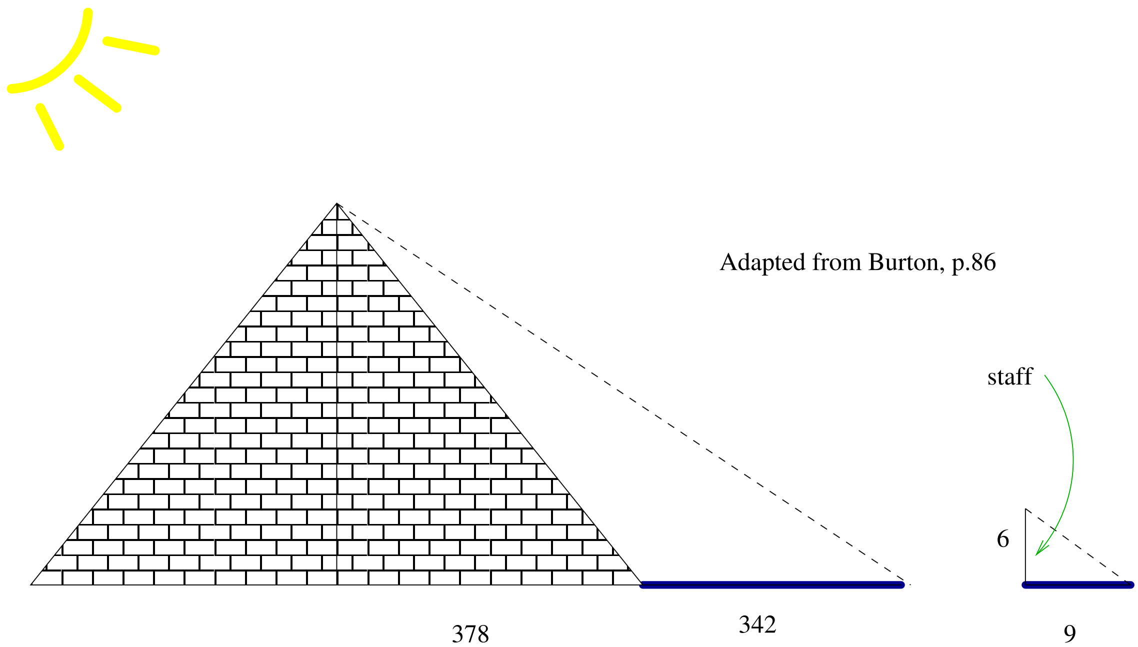
Thales (624–547 BCE) is often considered the first Greek mathematician in this tradition. Here are some of the theorems credited to him:

- The base angles of an isosceles triangle are equal. (I.e., if ΔABC is a triangle and $AB = AC$, then $m\angle ABC = m\angle ACB$.)
- Any angle inscribed in a semicircle is a right angle (you observed this in Math 409 in problem SA4).
- A circle is bisected by any diameter.

These are simple geometric results from a modern standpoint, but the important difference between them and earlier geometry was that Thales stated them as abstract observations about lines, circles, angles and triangles, rather than about counting oxen or measuring fields.

There is a legend that Thales impressed the Egyptians by determining the height of the Great Pyramid. (The square base could be measured directly, but not the height.) He placed his staff on the ground and measured the lengths of the shadows cast by the pyramid and by his staff.

[1] The source for this and much other material in thes notes is chapters 2–3 of D. Burton, *Burton's History of Mathematics: An Introduction*, 3rd edn., Wm. C. Brown Publishers, 1991.



(Presumably these units are in khet; see last week's notes.) In the figure, the number 378 is half the length of a side of the square base — this could be measured directly. So could 342 (the length of the shadow) and 6 and 9 the height of the staff and the length of its shadow).

Now, said Thales, we have two similar triangles, so the height h of the pyramid is given by the equation

$$\frac{h}{378+342} = \frac{6}{9}$$

which can be solved to give $h = 480$ khet.

Thales was able to realize that an abstract theorem about similar triangles could be applied to easily solve this practical problem. This is one of the first examples of modeling: solving a real-life problem by replacing it by a mathematical problem in order to be able to apply general theorems. Abstract mathematical knowledge can have concrete benefits!

Pythagoras (569–475 BCE) was a mystic whose religious beliefs included a strong mathematical component: "All is number" (by which they meant "integer"). He (or his followers) noticed that if you two take two strings made of the same material, one twice the length of the other, and pluck them, then the longer string will produce a sound an octave lower.[2] They noticed that other good-sounding musical intervals arose from pairs of strings whose lengths were in ratios of small integers.

Here's a question that the Pythagoreans might have asked: what pitch is half an octave? That is, we have two strings of lengths L and $2L$, and we want to determine the length M of a string that will produce a sound halfway in between. Since relative pitch is controlled by ratios, the lengths M and L must satisfy the equation

$$\frac{M}{L} = \frac{2L}{M}$$

Clearing denominators gives $M^2 = 2L^2$, so $M = \sqrt{2}L$. If you are a Pythagorean, this raises a natural question: what is $\sqrt{2}$? The Babylonians had found an excellent approximation[3] to $\sqrt{2}$, but the Pythagoreans were not interested in approximations; they wanted to understand the exact value. To their consternation, they discovered that $\sqrt{2}$ is in fact an irrational number! Here is their famous proof.

[2] We know today that musical pitch depends on frequency, and that doubling the length will halve the frequency.

[3] Accurate to five decimal places: see, e.g., this entry from mathematician John Baez's blog.

Theorem 1. $\sqrt{2}$ *is an irrational number. That is, there is no way to express it as a fraction* a/b*, where* a *and* b *are integers.*

Proof. Suppose that $\sqrt{2}$ is rational; that is, there exist positive[4] integers a, b such that

$$\sqrt{2} = a/b. \tag{1}$$

If we square both sides of equation (1) and clear denominators, we get

$$2b^2 = a^2. \tag{2}$$

One thing to notice from this equation is that $a > b$. Another observation is that a has to be even. If it were odd, then a^2 would be odd as well, and equation (2) would say that an even number (namely $2b^2$) equals an odd number, and that's impossible. Since a is even, we can write it as $a = 2c$, where c is another positive integer. If we substitute $a = 2c$ into equation (2), we get $2b^2 = a^2 = (2c)^2 = 4c^2$, which implies that

$$b^2 = 2c^2. \tag{3}$$

Looking at equation (3), we now see in turn that $b > c$ and that b has to be even—otherwise we would again have an even number equalling an odd number. So we may write $b = 2d$, where d is another positive integer Substituting $b = 2d$ into equation (3), we obtain $2c^2 = b^2 = (2d)^2 = 4d^2$, that is,

$$c^2 = 2d^2. \tag{4}$$

This looks just like equation (2). Have we gotten anywhere, or are we just chasing ourselves around in circles?

In fact, this process has to end. Otherwise, we will end up with a sequence of positive integers

$$a,\ b,\ c,\ d,\ e,\ \ldots,\ z,\ \alpha,\ \beta,\ \ldots,\ \omega,\ \aleph,\ \ldots$$

that keeps getting smaller and smaller (remember, $a > b$, $b > c$, and so on). But this can't happen! This means that eventually, it is impossible to continue the process. The only way to resolve this is to realize that our original assumption—namely, that we could find positive integers a and b such that $\sqrt{2} = a/b$—had to be false. Therefore, $\sqrt{2}$ is irrational. □

There is a widespread legend, almost certainly false, that the Pythagoreans were so upset over the existence of irrational numbers that they killed the discoverer of the theorem or the person who revealed it to the rest of the world. (Pythagoras left no writings himself, but he did acquire a mythical status after his death, with the result is that we have to rely on secondary sources, many of whom basically made up stories about him.)

This was one of the first proofs by contradiction. In a proof by contradiction, instead of deriving conclusions directly from your assumptions, you start by assuming that what you are trying to prove is false, and then show that this necessarily leads to something false—for example, that $2 + 2 = 6$, or that there exists an infinitely long decreasing sequence of positive integers. Proof by contradiction is an absolutely vital tool in mathematics!

For more reading:

- MacTutor: Greek mathematics
  http://www-history.mcs.st-andrews.ac.uk/Indexes/Greeks.html
- MacTutor: Biography of Thales
  http://www-history.mcs.st-andrews.ac.uk/Biographies/Thales.html
- MacTutor: Biography of Pythagoras
  http://www-history.mcs.st-andrews.ac.uk/Biographies/Pythagoras.html

[4] No one is disputing that $\sqrt{2} > 0$, so a and b have to be the same sign, and we might as well make them both positive instead of both negative.