The Mathematics of Networks (Chapter 7)

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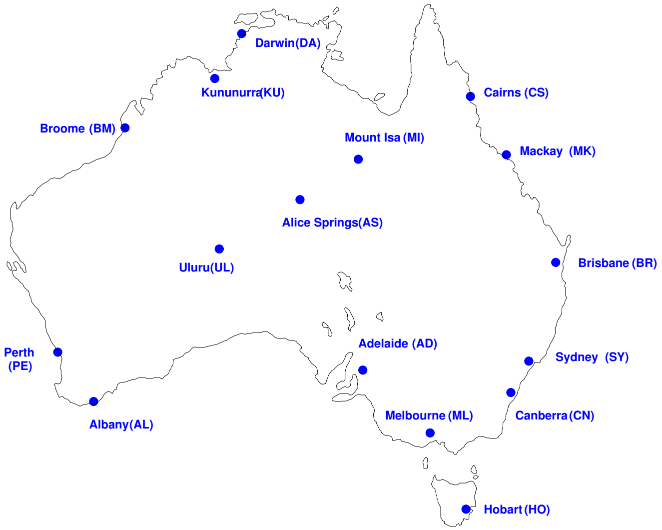
In other words, **What if we just want to connect all the vertices together in a network?**

- Roads, railroads
- Telephone lines
- Fiber-optic cable



Darwin(DA)
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Broome (BM)
Cairns (CS)
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Alice Springs(AS)
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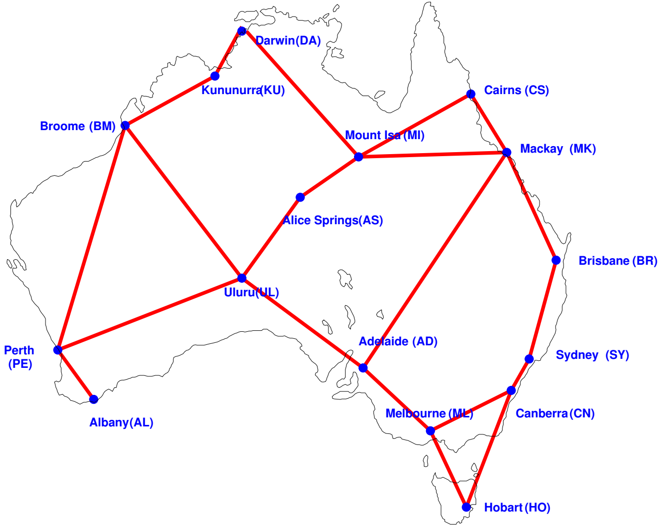
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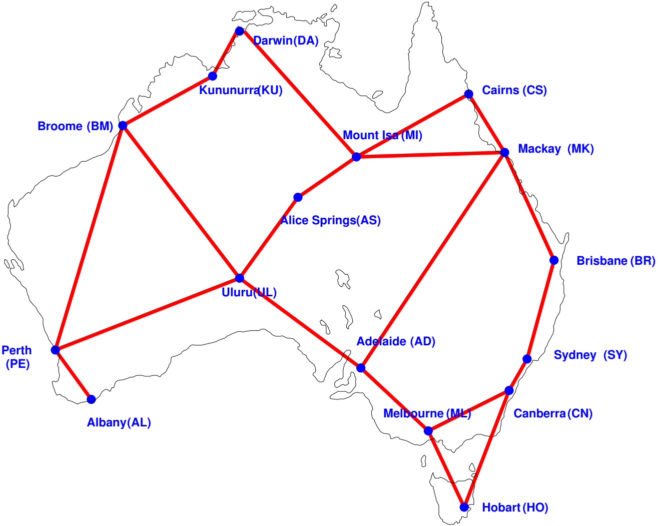
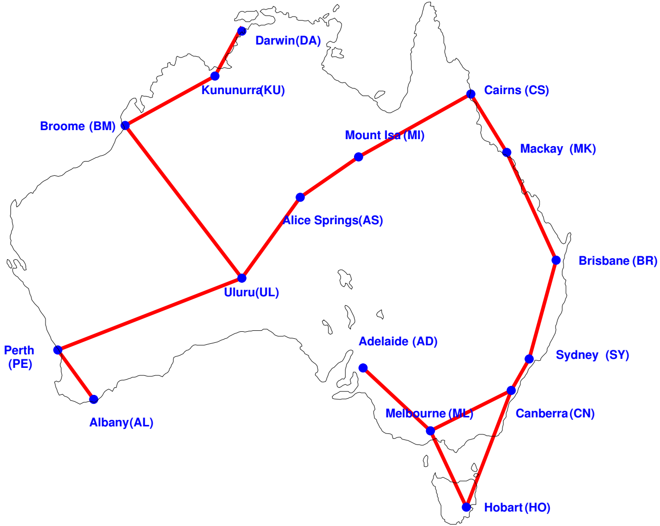
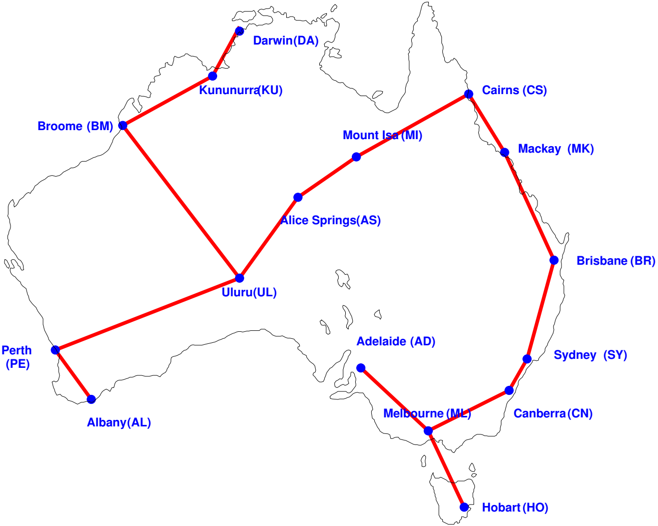


Too many edges!



Still too many edges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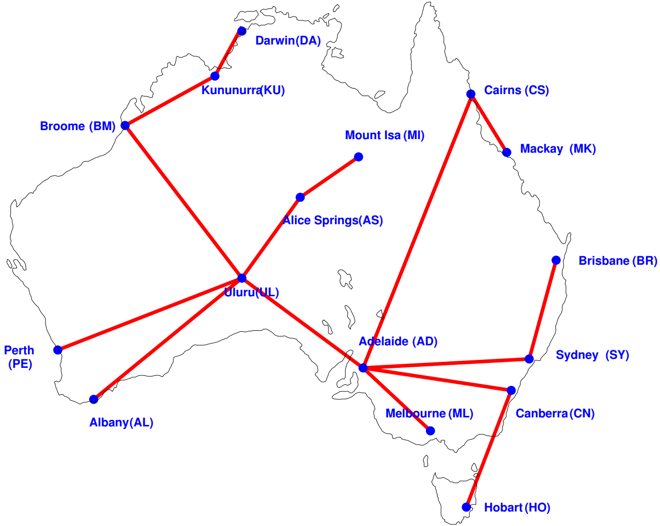
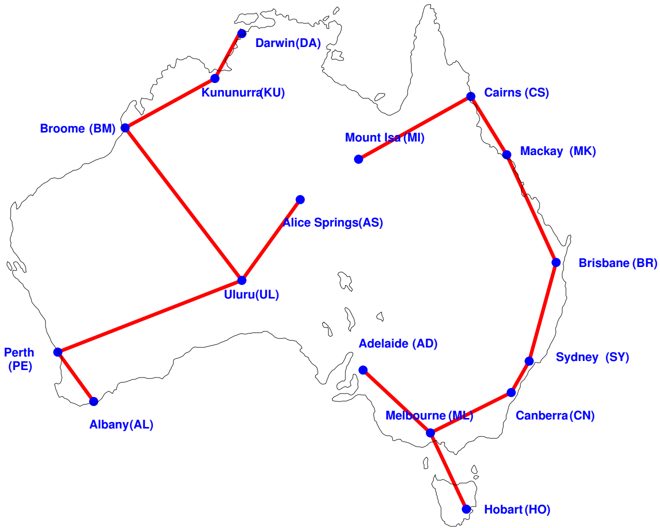
Just right



Another possibility



Not enough edges

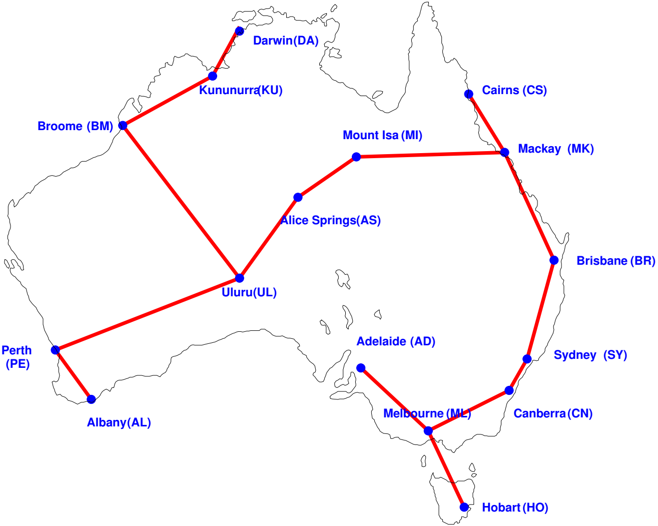
Networks and Spanning Trees

Definition: A **network** is a connected graph.

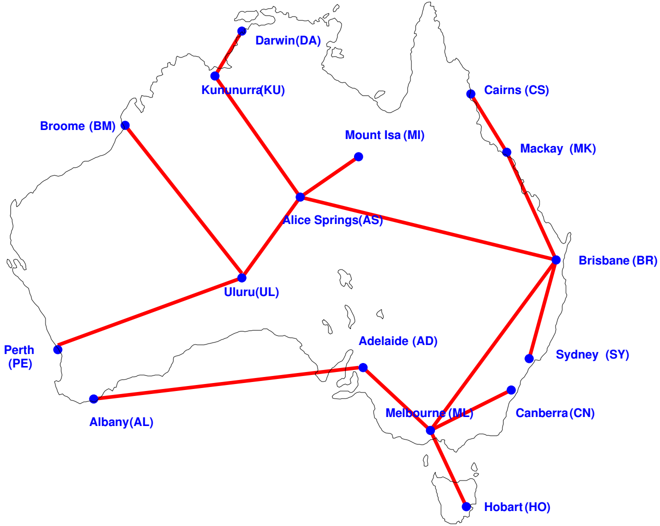
Definition: A **spanning tree** of a network is a subgraph that

1. connects all the vertices together; and
2. contains no circuits.

In graph theory terms, a spanning tree is a subgraph that is both **connected** and **acyclic**.



A spanning tree



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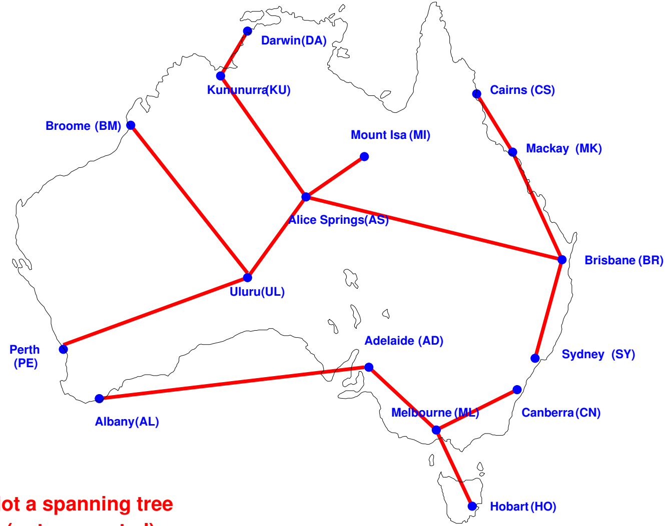
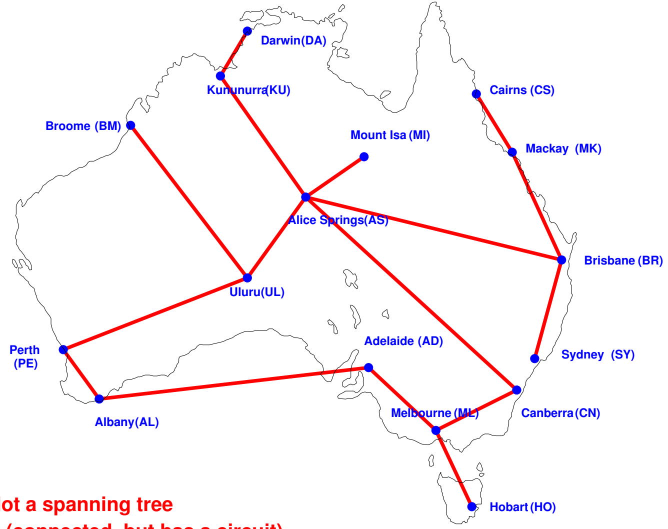
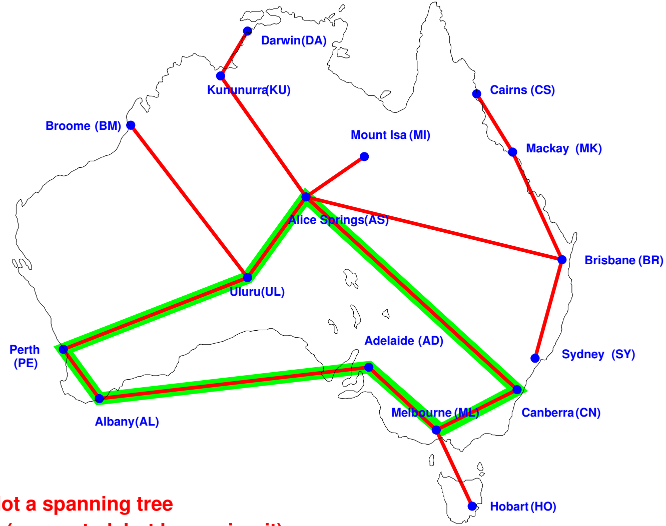


Not a spanning tree
(not connected)



Not a spanning tree
(connected, but has a circuit)



Not a spanning tree
(connected, but has a circuit)

The Number of Edges in a Spanning Tree

In a network with N vertices, how many edges does a spanning tree have?

The Number of Edges in a Spanning Tree

- Imagine starting with N isolated vertices and adding edges one at a time.

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- Imagine starting with N isolated vertices and adding edges one at a time.
- Each time you add an edge, you either
  - connect two components together, or
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- You have added exactly $N - 1$ edges.

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In a network with N vertices, every spanning tree has exactly N − 1 edges.

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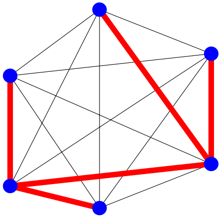
In a network with N vertices, every spanning tree has exactly $N - 1$ edges.

Must every set of $\mathbf{N - 1}$ edges form a spanning tree?

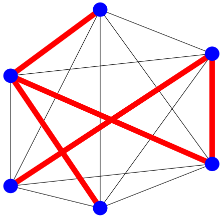
The Number of Edges in a Spanning Tree

Answer: **No.**

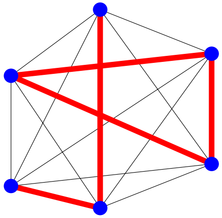
For example, suppose the network is K_4.



Spanning tree



Spanning tree

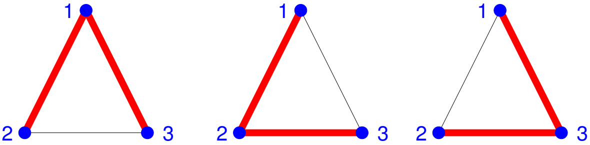


Not a spanning tree

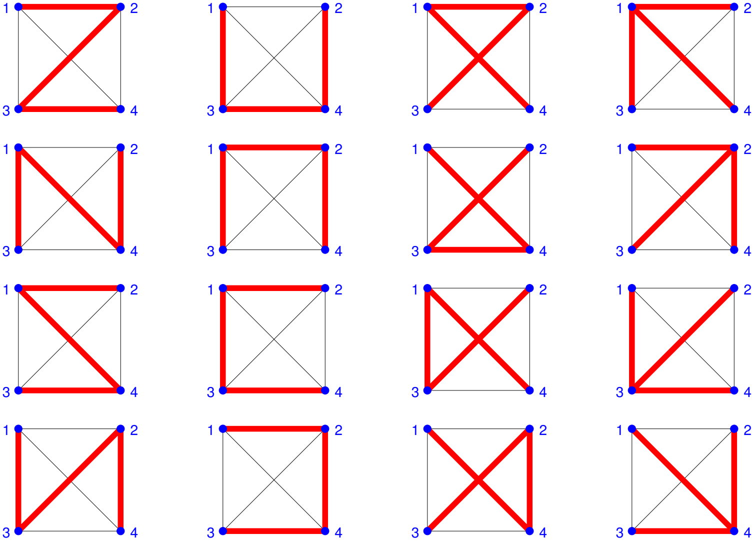
Spanning Trees in K_2 and K_3

K_2

K_3



Spanning Trees in K_4



Facts about Spanning Trees

Suppose we have a network with N vertices.

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Facts about Spanning Trees

4. In a network with N vertices and M edges,

$$M \geq N - 1$$

(otherwise it couldn't possibly be connected!) That is,

$$M - N + 1 \geq 0.$$

The number $M - N + 1$ is called the **redundancy** of the network, denoted by R.

Facts about Spanning Trees

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5. If $R = 0$, then the network is itself a tree.
If $R > 0$, then there are usually several spanning trees.

Counting Spanning Trees

We now know that every spanning tree of an N-vertex network has exactly $N - 1$ edges.

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Of course, this answer depends on the network itself.

Loops and Bridges

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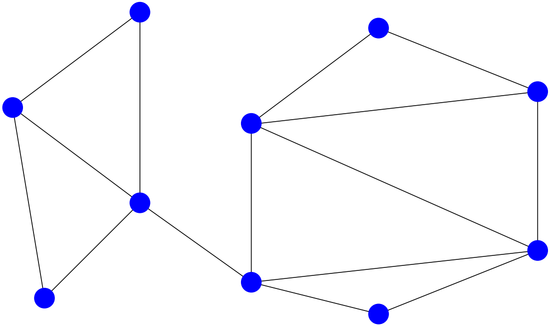

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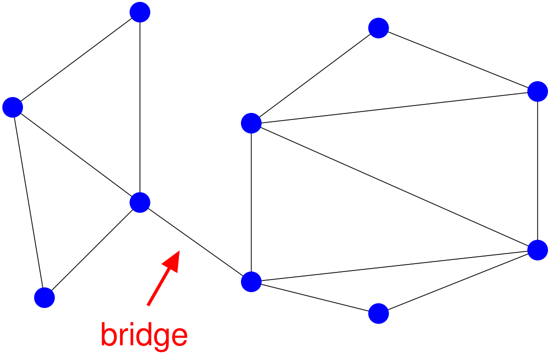

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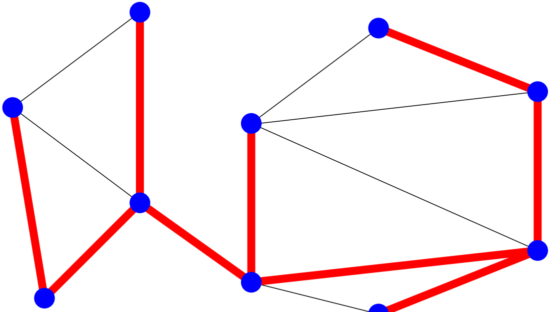

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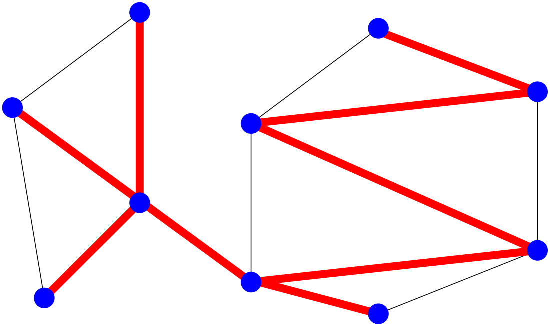

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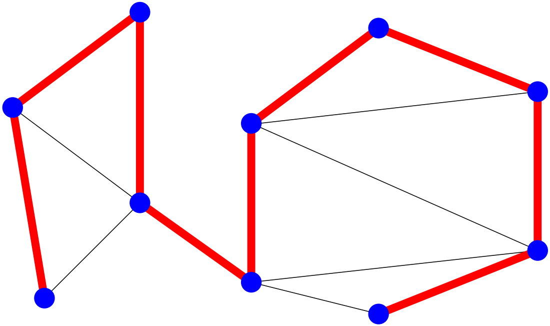

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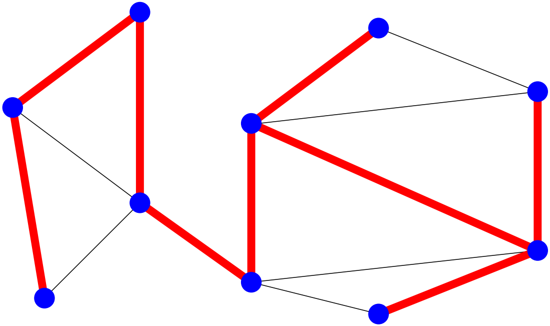

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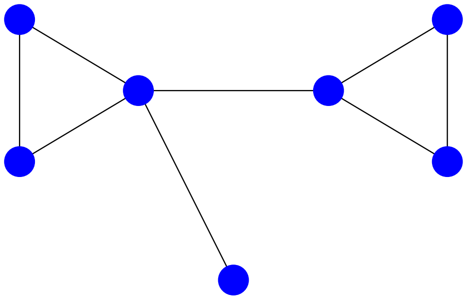

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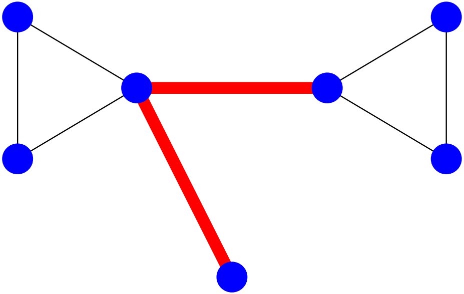

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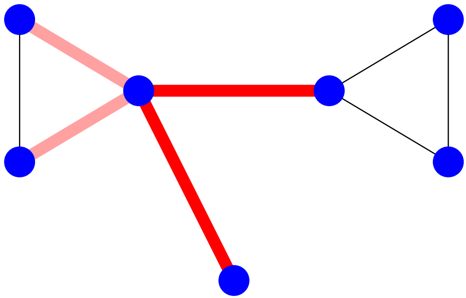

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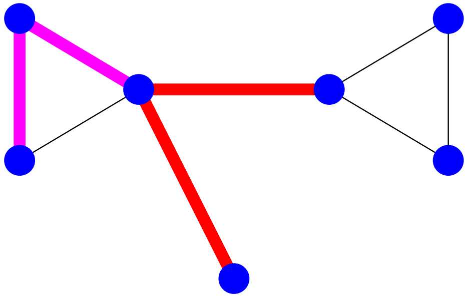

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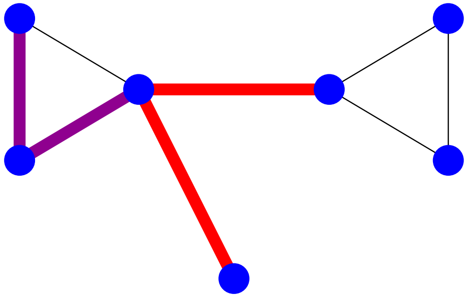

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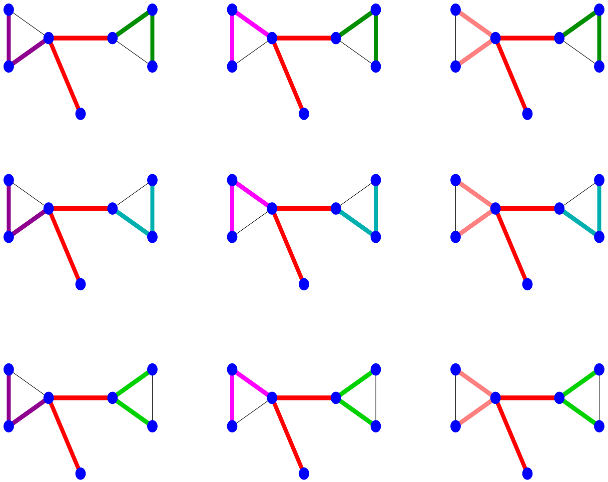

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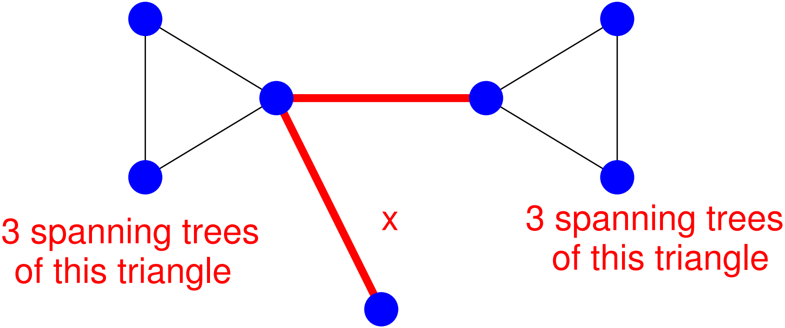
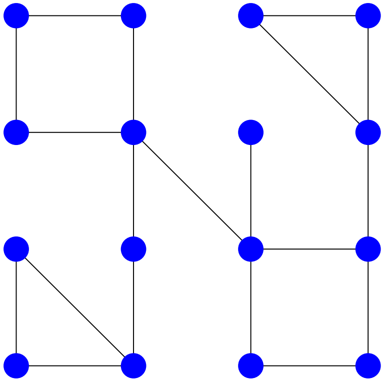


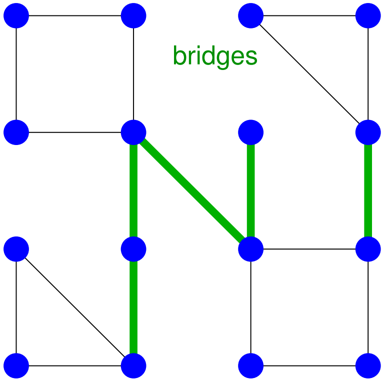
= 9 total spanning trees

Counting Spanning Trees



How many spanning trees does this network have?

Counting Spanning Trees



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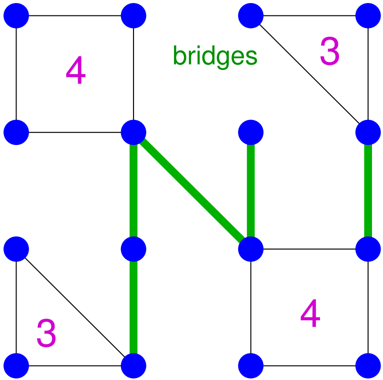

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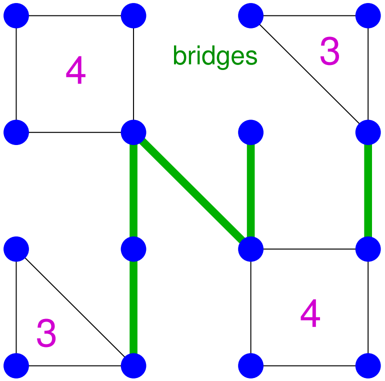
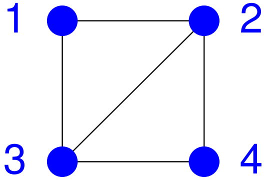


How many spanning trees does this network have?

Answer:
4 x 3 x 3 x 4 = **144.**

Counting Spanning Trees

If the graph has circuits that overlap, it is trickier to count spanning trees. For example:



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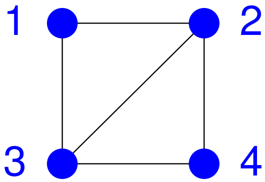

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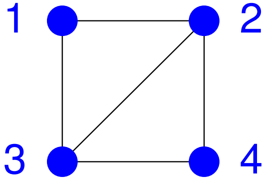

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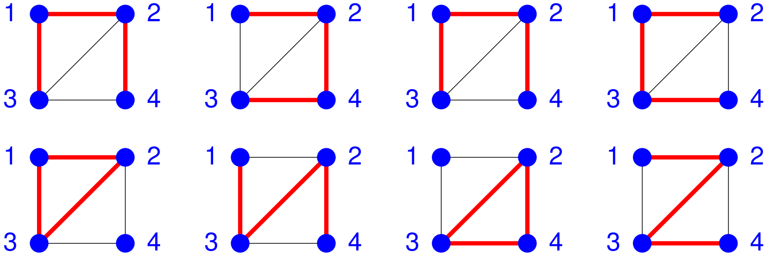
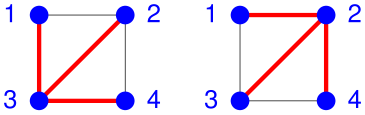
- List all the sets of three edges and cross out the ones that are not spanning trees.

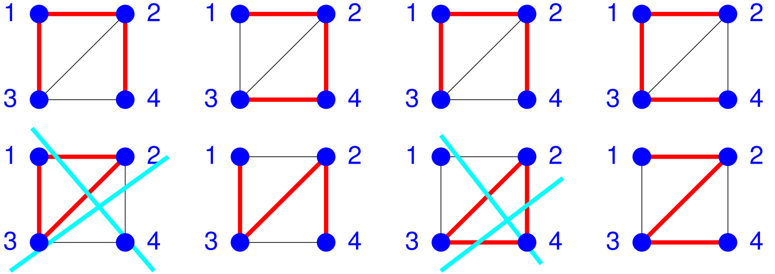
Counting Spanning Trees

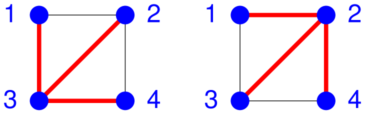




10 ways to select three edges

Counting Spanning Trees





10 ways to select three edges
2 are not trees
Total: 8 spanning trees

The Number of Spanning Trees of K_N (Not in Tannenbaum!)

Since K_N has N vertices, we know that every spanning tree of K_N has $N-1$ edges.

How many different spanning trees are there?

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We have already seen the answers for K_2, K_3, and K_4.

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Number of vertices (N)	Number of spanning trees in K_N
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3	3
4	16

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What's the pattern? ★

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The Number of Spanning Trees of K_N

Number of vertices (N)	Number of spanning trees in K_N
1	1
2	$1 = 2^0$
3	$3 = 3^1$
4	$16 = 4^2$
5	$125 = 5^3$
6	$1296 = 6^4$
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For example, K_{16} (the Australia graph!) has

$$16^{14} = 72,057,594,037,927,936$$

spanning trees.

(By comparison, the number of Hamilton circuits is "only"

$$15! \ = \ 1,307,674,368,000.)$$