Finer rook equivalence: Classifying Ding's Schubert varieties

Mike Develin (AIM)
Jeremy Martin (University of Minnesota)
Victor Reiner (University of Minnesota

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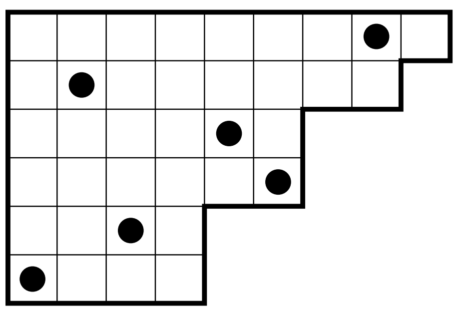
math.umn.edu/∼martin/math/pubs.html

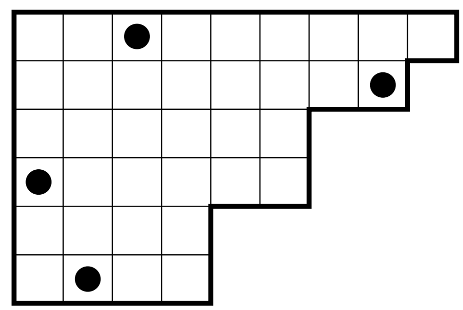
Rook theory

Let $\lambda = (0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ be a partition.

Defn A k-rook placement on λ consists of k squares of the Ferrers diagram (or "Ferrers board") of λ, no two in the same row or column.

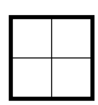
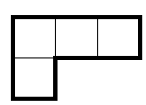


$\lambda = (4, 4, 6, 6, 8, 9)$



Defn $R_k(\lambda)$ = number of k-rook placements on λ

Defn λ, μ are rook-equivalent iff $R_k(\lambda) = R_k(\mu) \quad \forall k$.

Example $\lambda =$  $\mu =$ 

$$\begin{aligned} R_1(\lambda) &= R_1(\mu) = 4 \\ R_2(\lambda) &= R_2(\mu) = 2 \\ R_k(\lambda) &= R_k(\mu) = 0 \qquad \text{for } k > 2 \end{aligned}$$

Rook equivalence

Theorem (Foata–Schützenberger 1970)

Each rook-equivalence class contains a unique partition with distinct parts.

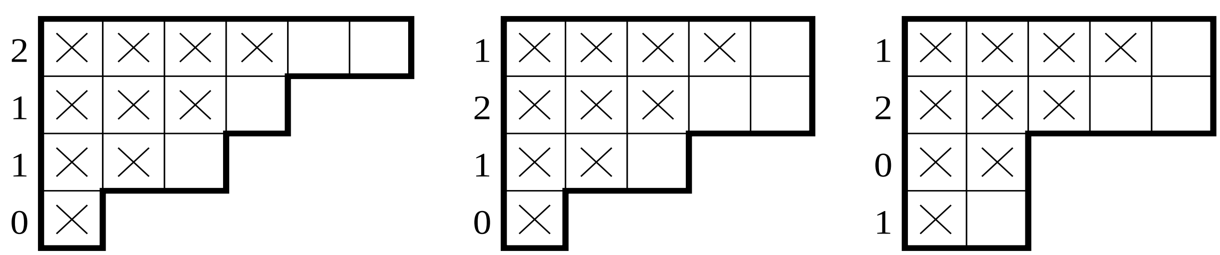
Theorem (Goldman–Joichi–White 1975)

Two partitions

$$\begin{aligned} \lambda &= (0 < \lambda_1 \leq \cdots \leq \lambda_n) \\ \mu &= (0 < \mu_1 \leq \cdots \leq \mu_n) \end{aligned}$$

are rook-equivalent iff $\{\lambda_i - i\}_{i=1}^{n} = \{\mu_i - i\}_{i=1}^{n}$ as multisets.

Example $GJW(\lambda) = \{0, 1, 1, 2\}$



q-counting maximal rook placements

Enumerate rook placements by an "inversion" statistic (generalizing inversions of permutations):

$$R_k(\lambda, q) \;=\; \sum_{k\text{-rook placements } \sigma} q^{\mathrm{inv}(\sigma)}$$

Theorem (Garsia–Remmel 1986)

(1) λ, μ are rook-equivalent iff they are q-rook equivalent.

(2) If $\lambda \;=\; (\lambda_1 \leq \cdots \leq \lambda_n)$, then up to a factor of q,

$$R_n(\lambda, q) \;=\; \prod_{i=1}^{n} [\lambda_i - i + 1]_q$$

where $[m]_q \;=\; \frac{q^m - 1}{q-1} \;=\; 1 + q + q^2 + \cdots + q^{m-1}$.

Observations

(1) If $\lambda_i < i$ for some i (that is, λ does not contain a staircase), then $R_n(\lambda, q) = 0$.

(2) If $\lambda_n = n$, then λ is rook-equivalent to $(\lambda_1, \ldots, \lambda_{n-1})$.

Ding's Schubert varieties

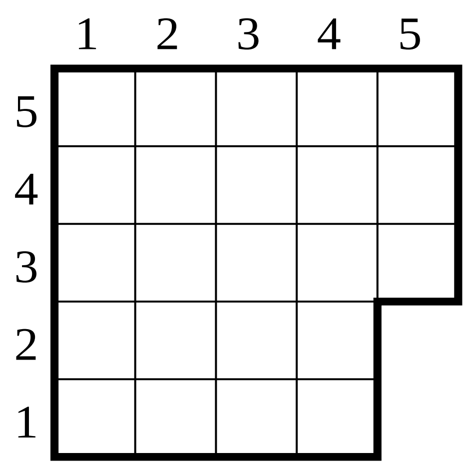
- $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n = m), \qquad \lambda_i \geq i$ (λ contains a staircase)

- $\mathbb{C}^0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^m$: standard flag

Defn $X_\lambda = \left\{ \begin{array}{c} \text{flags } 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^m \ : \\ \forall i : \ \dim_{\mathbb{C}} V_i = i, \quad V_i \subset \mathbb{C}^{\lambda_i} \end{array} \right\}.$

- X_λ is a Schubert variety X_w in a type-A partial flag manifold Y

Example $\lambda = (4,4,5,5,5) \quad w = 43521 \in S_5$



- w is 312-avoiding; in particular X_w is smooth

- $[X_w] \in H^*(Y)$ is a Schubert polynomial indexed by the dominant permutation $w_0 w w_0$

The cohomology ring of X_λ

Defn $R^\lambda \;:=\; H^*(X_\lambda\,;\,\mathbb{Z}) \;=\; \bigoplus_i H^{2i}(X_\lambda\,;\,\mathbb{Z})$

(because X_λ has no torsion or odd-dimensional cohomology)

Theorem (Ding)

$$\sum_i q^i \ \mathrm{rank}_{\mathbb{Z}}\, H^{2i}(X_\lambda) \quad = \quad R_n(\lambda, q).$$

Theorem (Gasharov–Reiner)

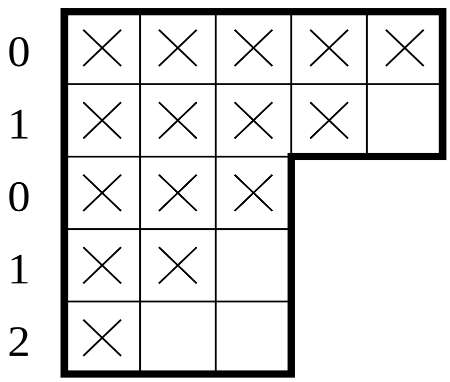
$$H^*(X_\lambda) \;\;\cong\;\; \mathbb{Z}[x_1, \ldots, x_n]/I_\lambda$$

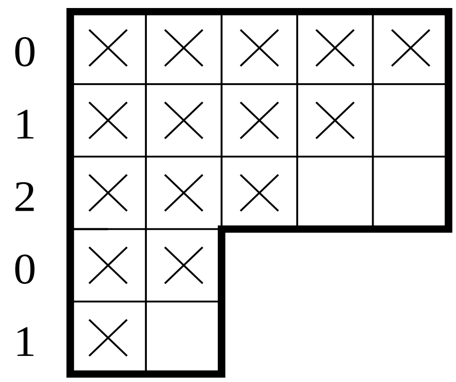
where $I_\lambda = \langle h_{\lambda_i - i+1}(x_1, \ldots, x_i) :\ 1 \le i \le n\rangle$.

Observation If $\lambda_i < i$ for some i (that is, λ does not contain a staircase), then $X_\lambda = \emptyset$.

Trivial isomorphisms among the X_λ's

Observation Suppose that $\lambda_i = i$ for some i:





$$X_\lambda = \{V_\bullet : V_1 \subset V_2 \subset V_3 = \mathbb{C}^3 \subset V_4 \subset \mathbb{C}^5\} \cong Fl_3 \times Fl_2$$
$$X_\mu = \{V_\bullet : V_1 \subset V_2 = \mathbb{C}^2 \subset V_3 \subset V_4 \subset \mathbb{C}^5\} \cong Fl_2 \times Fl_3$$

$$\begin{aligned} R^\lambda &= \mathbb{Z}[x_1, \ldots, x_5] \,/\, \langle h_3(1), h_2(2), h_1(3), h_2(4), h_1(5) \rangle \\ &= \mathbb{Z}[x_1, x_2, x_3] \,/\, \langle e_1, e_2, e_3 \rangle \bigotimes_{\mathbb{Z}} \mathbb{Z}[x_4, x_5] \,/\, \langle e_4, e_5 \rangle \end{aligned}$$

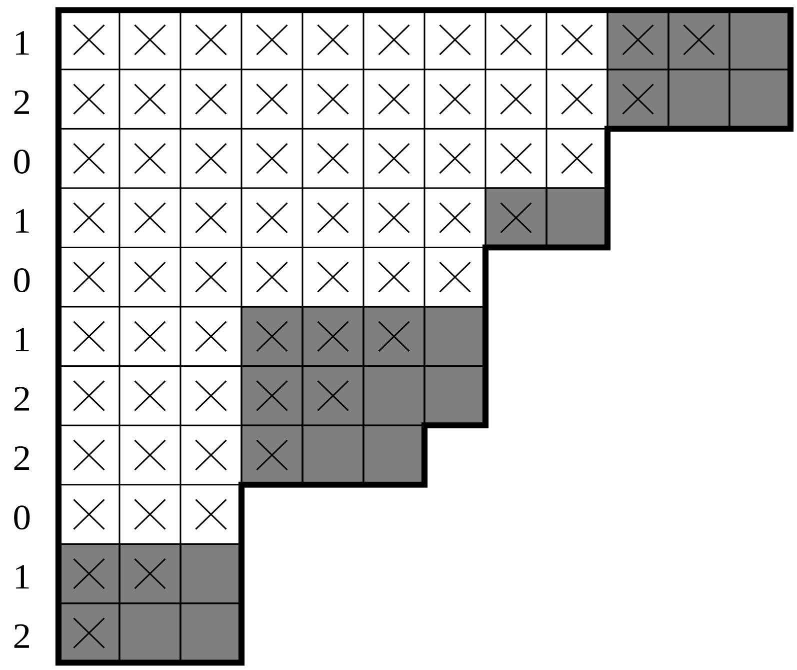
$$R^\mu = \mathbb{Z}[x_1, x_2] \,/\, \langle e_1, e_2 \rangle \bigotimes_{\mathbb{Z}} \mathbb{Z}[x_3, x_4, x_5] \,/\, \langle e_3, e_4, e_5 \rangle$$

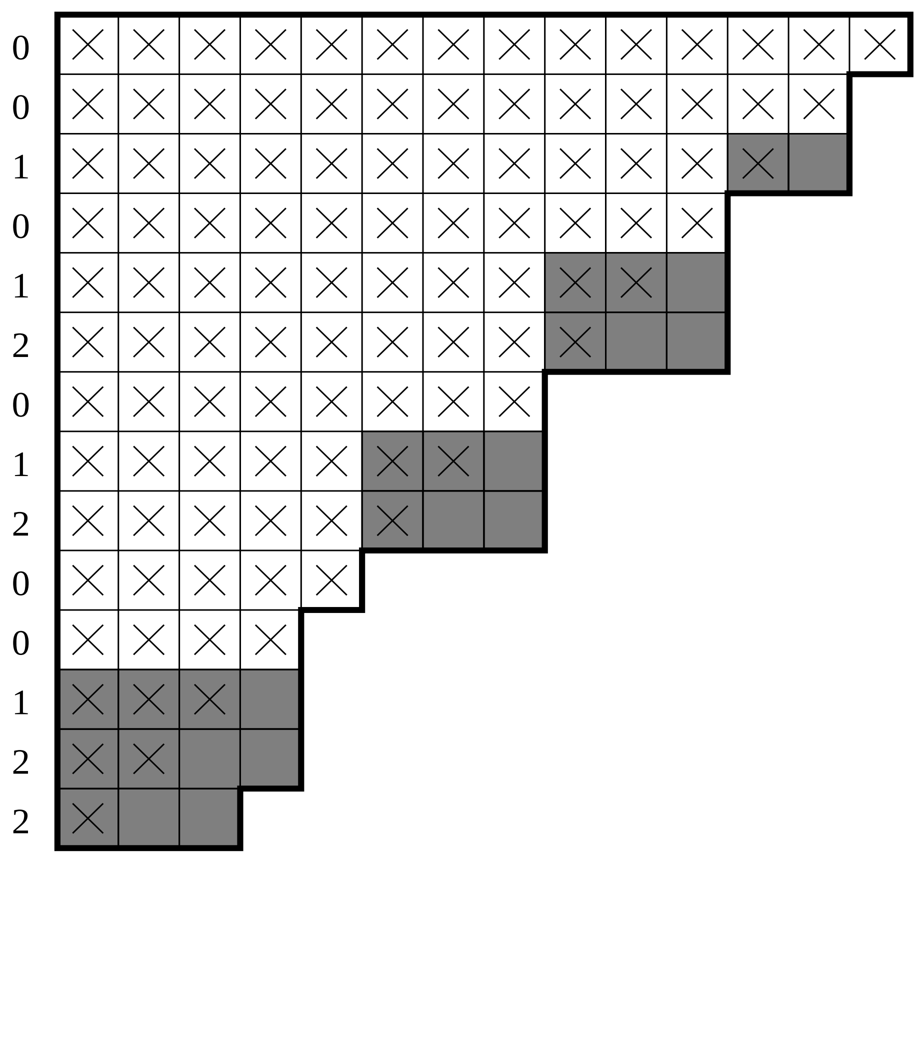
In general,

$$X_\lambda \cong \prod_j X_{\lambda^{(j)}}, \qquad R^\lambda \cong \bigotimes_j R^{\lambda^{(j)}}$$

where $\lambda^{(j)}$ are the <u>indecomposable components</u> of λ.

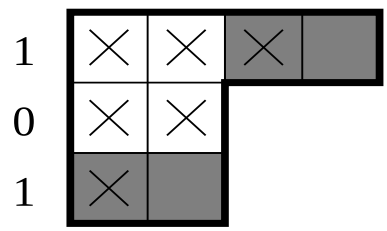
Fine rook equivalence

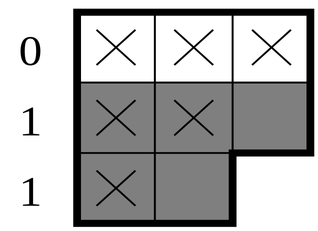




Rook equivalence is not enough

$\lambda = (2, 2, 4)$ $\mu = (2, 3, 3)$





$R^\lambda \cong \mathbb{Z}[x, y] \,/\, \langle x^2, y^2 \rangle$ $R^\mu \cong \mathbb{Z}[s, t] \,/\, \langle s^2, st + t^2 \rangle$

λ and μ are rook-equivalent, and both cohomology rings have Poincaré series $1 + 2q + q^2$. But consider

$$\begin{aligned} \{\text{primitive } f \in R^\lambda_1 \ : \ f^2 = 0\} &= \{x, y\}, \\ \{\text{primitive } f \in R^\mu_1 \ : \ f^2 = 0\} &= \{s, s + 2t\}. \end{aligned}$$

The former is a $\mathbb{Z}$-basis for $H^1(X_\lambda)$, while the latter is not a $\mathbb{Z}$-basis for $H^1(X_\mu)$. Therefore $\boldsymbol{X_\lambda \ncong X_\mu}$.

In fact, $R_\lambda \cong \mathbb{Z}[x] \,/\, \langle x \rangle \ \otimes \ \mathbb{Z}[y] \,/\, \langle y \rangle$, while R_μ does not decompose as a tensor product of smaller rings.

The main classification theorem

Theorem (D–M–R) For partitions λ and μ with indecomposable components

$$\lambda^{(1)}, \ldots, \lambda^{(r)}, \qquad \mu^{(1)}, \ldots, \mu^{(s)},$$

the following are equivalent:

(1) The multisets $\{\lambda^{(i)}\}_{i=1}^{r}$ and $\{\mu^{(i)}\}_{i=1}^{s}$ are identical.

(2) $X_\lambda \cong X_\mu$ as algebraic varieties.

(3) $H^*(X_\lambda; \mathbb{Z}) \cong H^*(X_\mu; \mathbb{Z})$ as graded rings.

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(1) $\Longrightarrow$ (2): Follows from trivial isomorphisms.

(2) $\Longrightarrow$ (3): Immediate.

- The hard part is (3) $\Longrightarrow$ (1).

Overview of the proof

Main idea: In order to recover $\lambda_1, \ldots, \lambda_n$ from the structure of $R^\lambda = H^*(X_\lambda)$ as a graded $\mathbb{Z}$-algebra ...

... study nilpotence orders of linear forms.

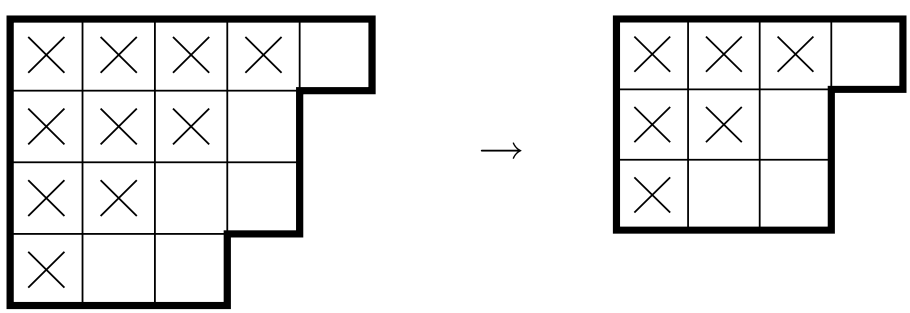
Defn The <u>nilpotence order</u> of a homogeneous element $f \in R^\lambda$ is

$$\mathrm{nilpo}(f) \;=\; \min\left\{n \in \mathbb{N} : \; f^n = 0\right\}.$$

Proposition If λ is indecomposable, then

$$\min\left\{\mathrm{nilpo}(f) : \; f \in R_1^\lambda\right\} \;=\; \lambda_1.$$

Proposition $R^\lambda \,/\, \langle x_1 \rangle \cong R^\mu$, where μ is the partition obtained by "peeling off" the leftmost column and bottom row of λ:



So we can just read off λ from the structure of R^λ by taking successive quotients by linear forms of appropriate nilpotence order, right?

Well...

Good and bad nilpotents

Problem Identify a λ_1-nilpotent linear form f with

$$H^*(X^\lambda)/\langle f\rangle \;\cong\; H^*(X^\lambda)/\langle x_1\rangle$$

(for instance, $f = x_1$),

$$\boxed{\text{independently of the presentation } H^*(X^\lambda) \cong R^\lambda/I_\lambda.}$$

Theorem For λ indecomposable and

$$k = \lambda_1 = \lambda_2 = \cdots = \lambda_m < \lambda_{m+1},$$

the λ_1-nilpotents in R_1^λ are exactly the following:

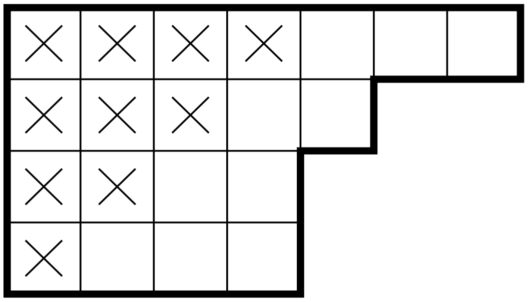
$x_1,\ x_2,\ \ldots,\ x_m$	(in all cases)
$x_1 + \ldots + x_m$	(iff $m = k-1$)
$x_1 + \ldots + x_m + 2x_{m+1}$	(iff $m = k-1$, $\lambda_k = k+1$, and k is even)

- The "good" nilpotents $x_1, \ldots, x_m$ can be distinguished intrinsically from the "bad" ones.

- Necessary to show that R^λ has a unique maximal tensor product decomposition into the $R^{\lambda^{(i)}}$'s.

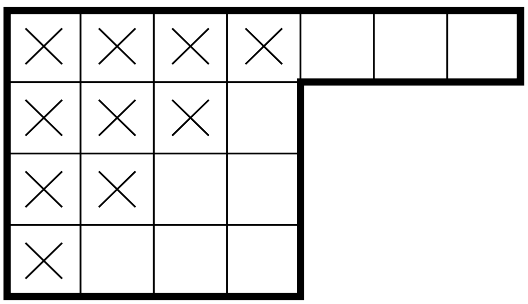
(This is probably not true for standard graded $\mathbb{Z}$-algebras in general!)

Partitions $\boldsymbol{\lambda}$ | **$\boldsymbol{\lambda}_1$-nilpotents in $\boldsymbol{R}_1^{\boldsymbol{\lambda}}$**



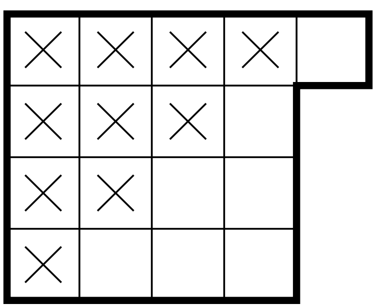
$k = 4$, $m = 2$

$x_1, \;\; x_2, \;\; x_3$



$k = 4$, $m = 3$

$x_1, \;\; x_2, \;\; x_3,$
$x_1 + x_2 + x_3$

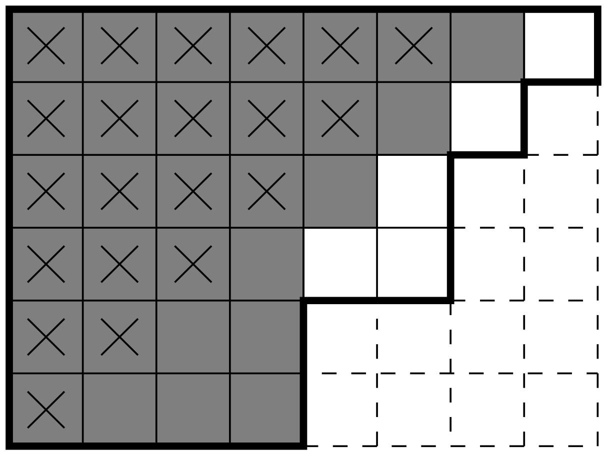


$k = 4$, $m = 3$, $\lambda_4 = 5$

$x_1, \;\; x_2, \;\; x_3,$
$x_1 + x_2 + x_3,$
$x_1 + x_2 + x_3 + 2x_4$

Gröbner bases, cores and stickiness

Fact If $\mu \subset \lambda$, then $X_\mu \hookrightarrow X_\lambda$ and $R^\lambda \twoheadrightarrow R^\mu$.



$$\underset{\text{core of } \lambda}{(4,4,4,5,6,7)} \quad \subset \quad \lambda = (4,4,6,6,7,8) \quad \subset \quad \underset{\text{rectangle}}{(8,8,8,8,8,8)}$$

- If you want to prove that $f = 0$ in R^λ …
  …replace λ with a larger rectangle.
- If you want to prove that $f \neq 0$ in R^λ …
  …replace λ with its core.

Proposition If λ is indecomposable and its own core, then the generators of I_λ can be manipulated to produce a Gröbner basis in which the variables $x_{\lambda_1}, \ldots, x_n$ are "sticky".

I.e., if $\lambda_1 \leq j \leq n$ and $f \in R^\lambda$ involves x_j, then all partial Gröbner reductions of f involve x_j.

Questions for further study

1. Poset rook equivalence

When are two rook-placement posets RP_λ, RP_μ isomorphic?

- Strictly stronger than rook equivalence
- Strictly weaker than $X_\lambda \cong X_\mu$

2. Nilpotence and the Schubert variety

- What do all these (Gröbner) calculations say about the (enumerative) geometry of X_λ?
- Nilpotence $\iff$ self-intersection numbers?

3. Other Schubert varieties

- Find a presentation for $H^*(X_w\,;\,\mathbb{Z})$, where $X_w \subset GL_n/B$
- Can these be used to classify arbitrary X_w up to isomorphism?