Simplicial Effective Resistance and Tree Enumeration

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The Menu

1. **Spanning Trees and How To Count Them**
   you probably know most of this part already

2. **Resistor Networks**
   you might have seen this in a course on graph theory

3. **Simplicial Trees**
   recent work of Art Duval, Carly Klivans, and myself

4. **Simplicial Networks and Applications**
   finally, the new stuff

1. Spanning Trees and How To Count Them

Spanning Trees and the Matrix-Tree Theorem

Let $G = (V, E)$ be a connected loopless graph, $V = [n]$.

- **Spanning tree** T: maximal acyclic edge set (or subgraph)
- Every spanning tree has $n - 1$ edges
- $\mathcal{T}(G)$ = set of spanning trees; $\tau(G) = |\mathcal{T}(G)|$
- Lovely formulas: $\tau(K_n) = n^{n-2}$ ("Cayley"), $K_{p,q}$, Q_n, ...

Matrix-Tree Theorem: Let $L = L(G)$ be the **Laplacian matrix**

$$L = [\ell_{ij}]_{i,j=1}^{n} \qquad \ell_{ij} = \begin{cases} \deg(i) & \text{if } i = j, \\ -|E_{i,j}| & \text{if } i \neq j \end{cases}$$

where $E_{i,j}$ = set of edges with endpoints i, j. Then

$$\tau(G) \;=\; \det L_{V\setminus i, V\setminus i} \;=\; \frac{\prod \text{nonzero eigenvalues of } L}{n}.$$

One Proof of the Matrix-Tree Theorem

Orient each edge $e = ij$ as $\overrightarrow{ij}$; i is the *tail* and j is the *head*. The **signed incidence matrix** of G is

$$\partial = [\partial_{ie}]_{i \in V,\ e \in E} \qquad \partial_{i,e} = \begin{cases} 1 & \text{if } i = \text{head}(e), \\ -1 & \text{if } i = \text{tail}(e), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$L = \partial \partial^T, \qquad L_{V \setminus i, V \setminus i} = \partial_{V \setminus i, E} \partial^{tr}_{V \setminus i, E}$$

and by the Binet-Cauchy identity

$$\det L^{ii} = \sum_{A \subseteq E:\ |A| = n-1} \det(\partial_{V \setminus i, A}) \det(\partial^t_{V \setminus i, A}) = \sum_A \det(\partial_{V \setminus i, A})^2$$

and the summand is $(\pm 1)^2$ if A is a tree, 0 otherwise.

Weighted Tree Enumeration

Assign each edge a weight x_e. The **weighted Laplacian** is

$$\hat{L} = [\hat{\ell}_{ij}]_{i,j=1}^{n} \qquad \ell_{ij} = \begin{cases} \sum_{e \in E_i} x_e & \text{if } i = j, \\ -\sum_{e \in E_{i,j}} x_e & \text{if } i \neq j. \end{cases}$$

Weighted Matrix-Tree Theorem

$$\sum_{T \in \mathcal{T}(G)} \prod_{e \in T} x_e = \det \hat{L}_{V \setminus i, V \setminus i}.$$

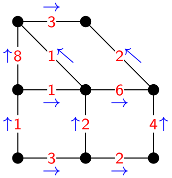
Application: Introducing indeterminates $\{x_i : \; i \in V\}$ and setting $x_{ij} = x_i x_j$ can recover formulas like Cayley–Prüfer:

$$\sum_{T \in \mathcal{T}(G)} \prod_{v=1}^{n} x_i^{\deg_T(i)} = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}.$$

2. Resistor Networks

Resistor Networks

A **[resistor] network** $N = (V, E, \mathbf{r})$ is a connected, undirected* graph (V, E) together with positive **resistances** $\mathbf{r} = (r_e)_{e \in E}$.



State of N:

currents $\mathbf{i} = (i_e)_{e \in E}$

voltages $\mathbf{v} = (v_e)_{e \in E}$

Ohm's law $\quad i_e r_e = v_e \quad (\forall e \in E)$

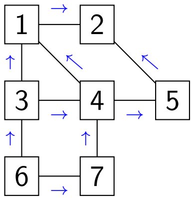
Kirchhoff's current law $\quad \sum_{e \in E^{\text{in}}(x)} i_e - \sum_{e \in E^{\text{out}}(x)} i_e = 0 \quad (\forall x \in V)$

Kirchhoff's voltage law $\quad \sum_{\vec{e} \in C} v_e = 0 \quad (\forall \text{ cycle } C)$

Every voltage comes from a **potential** $(p_i)_{i \in V}$ via $v_{\overrightarrow{ij}} = p_j - p_i$

*Edges oriented for reference purposes only.

Kirchhoff's Laws and the Incidence Matrix



$$
\begin{array}{cc}
\boxed{\partial} & \begin{array}{ccccccccc} 12 & 31 & 41 & 52 & 34 & 45 & 63 & 74 & 67 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{pmatrix}
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\end{array}
$$

KCL: $\mathbf{i} \in \ker \partial = \text{nullspace}(\partial)$

- Currents are **flows**

KVL: $\mathbf{v} \in (\ker \partial)^{\perp} = \text{rowspace}(\partial)$

- Voltages are **cuts**

Effective Resistance

Idea: Attach a **current generator**: an edge $\mathbf{e} = \overrightarrow{xy}$ with current $i_{\mathbf{e}}$, then look for currents and voltages satisfying OL, KCL, KPL.

Dirichlet principle The state of the system is the unique minimizer of "total energy" $\sum_e v_e i_e$ subject to OL, KCL, KPL.

Rayleigh principle As far as the external world is concerned, the system is equivalent to a single edge $\mathbf{e}$ with resistance

$$R_{\mathbf{e}}^{\text{eff}} = R_{xy}^{\text{eff}} = \frac{p_y - p_x}{c_{\mathbf{e}}}.$$

(the **effective resistance** of $\mathbf{e}$).

To calculate $R_{\mathbf{e}}^{\text{eff}}$: assign $\mathbf{e}$ unit current, find $\mathbf{v}$ and $\mathbf{i}$ minimizing energy. Then $R_{\mathbf{e}}^{\text{eff}} = \mathbf{v}_{\mathbf{e}}$.

Effective Resistance and Tree Counting

Theorem [Thomassen 1990]
Let $N = (V, E, \mathbf{r})$ be a network and $e = xy \in E$.

- If $\mathbf{r} \equiv 1$, then

$$R_{xy}^{\text{eff}} = \frac{\tau(G/xy)}{\tau(G)} = \Pr[\text{random spanning tree contains } xy]$$

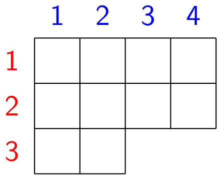
- Generalization for arbitrary resistances:

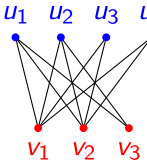
$$R_{xy}^{\text{eff}} = \frac{\hat{\tau}(G/xy)}{\hat{\tau}(G)} = \frac{\displaystyle\sum_{T \in \mathcal{T}(G/xy)} \prod_{e \in T} r_e^{-1}}{\displaystyle\sum_{T \in \mathcal{T}(G)} \prod_{e \in T} r_e^{-1}}.$$

Combinatorial application: weighted tree enumeration!

Application: Ferrers Graphs

The **Ferrers graph** G_λ of a partition λ has vertices corresponding to the rows and columns of λ, and edges corresponding to squares.



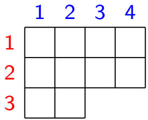


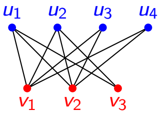
Here $\lambda = (4, 4, 2)$, $\lambda' = (3, 3, 2, 2)$, $n = 3 = \ell(\lambda)$, $m = 4 = \ell(\lambda')$.

Define a degree-weighted tree enumerator

$$\hat{\tau}(G) = \sum_{T \in \mathcal{T}(G_\lambda)} \prod_{i=1}^{m} {x_i}^{\deg_T(u_i)} \prod_{j=1}^{n} {y_j}^{\deg_T(v_j)}$$

Application: Ferrers Graphs





Theorem (Ehrenborg and van Willigenburg, 2004):

$$\hat{\tau}(G_\lambda) = x_1 \cdots x_m \, y_1 \cdots y_n \prod_{i=2}^{n} (y_1 + \cdots + y_{\lambda_i}) \prod_{j=2}^{n} (x_1 + \cdots + x_{\lambda'_j})$$

(Proof sketch: Find effective resistance of a corner of λ; induct.)

In the example above,

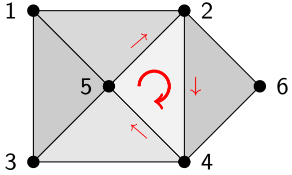
$$\begin{aligned} \hat{\tau}(G_\lambda) &= x_1 x_2 x_3 x_4 \, y_1 y_2 y_3 \\ &\times (y_1 + y_2 + y_3)(y_1 + y_2)^2 (x_1 + x_2 + x_3 + x_4)(x_1 + x_2) \end{aligned}$$

and in particular $\tau(G_\lambda) = 3 \cdot 2^2 \cdot 4 \cdot 2$.

3. Simplicial Trees

Simplicial Complexes

- **Geometric simplicial complex**: family of simplices (points, line segments, triangles, tetrahedra, ...) attached along faces
- **Combinatorial simplicial complex:** $\Delta \subseteq 2^V$ such that $\sigma \in \Delta,\ \rho \subseteq \sigma \implies \rho \in \Delta$



$$\langle 125, 135, 245, 345, 246 \rangle$$
$$= \{125, 135, 245, 345, 246,\\ 12, 13, 15, 24, 25, 26, 34, 35, 45, 46,\\ 1, 2, 3, 4, 5, 6,\quad \emptyset\}$$

- **Facets** $\Phi = \Phi(\Delta) =$ maximal faces
- Assume Δ^d **pure**: $|\phi| = d + 1$ for all facets ϕ

Boundary Maps and Homology Groups

Boundary of a k-simplex $\sigma = (v_0 < v_1 < \cdots < v_k)$:

$$\partial_k(v_0 < v_1 < \cdots < v_k) = \sum_{i=0}^{k} (-1)^i (v_0 \cdots \widehat{v_i} \cdots v_k)$$

Extending linearly gives a map

$$\partial_k : C_k(\Delta; R) \to C_{k-1}(\Delta; R)$$

where $C_k(\Delta; R) =$ linear combos of k-simplices ($R = \mathbb{R}$ or $\mathbb{Z}$)

- **Key fact:** $\partial_k \circ \partial_{k+1} = 0$.
- **Aha moment:** $\partial_1 =$ signed incidence matrix of graph $\Delta^{(1)}$

Boundary Maps and Homology Groups

The **simplicial chain complex** is

$$0 \to C_d(\Delta; R) \xrightarrow{\partial_d} C_{d-1}(\Delta; R) \to \cdots$$
$$\to C_1(\Delta; R) \xrightarrow{\partial_1} C_0(\Delta; R) \xrightarrow{\partial_0} C_1(\Delta; R) \to 0.$$

- $\partial_k \partial_{k+1} = 0$ implies $\ker \partial_k \supseteq \operatorname{im} \partial_{k+1}$
- **(reduced simplicial) homology:** $H_k(\Delta; R) = \ker \partial_k / \operatorname{im} \partial_{k+1}$
- Homology groups are topological invariants of Δ
- Over $\mathbb{Z}$: $H_k = \mathbb{Z}^{b_k} \oplus T_k$
  - b_k = *Betti number*: counts k-dimensional holes
  - T_k = *torsion group*: finite, measures nonorientability
- Over $\mathbb{R}$: $H_k = \mathbb{R}^{b_k}$

Spanning Trees of Simplicial Complexes

A **spanning tree** of Δ^d is a subcomplex $\Upsilon \subset \Delta$ such that:

1. Υ contains all non-maximal faces (**spanning**)
2. $H_d(\Upsilon; \mathbb{R}) = 0$ (**acyclic**)
3. $H_{d-1}(\Upsilon; \mathbb{R}) = 0$ (**connected**) $\iff$ $H_{d-1}(\Upsilon; \mathbb{Z})$ finite

Examples:

- $d = 1$: standard definition of spanning tree of a graph
- $\Delta =$ simplicial sphere: remove a facet
- $d = 2$: regard Δ as bubble wrap — pop all the bubbles but don't tear the bottom sheet

Counting Simplicial Spanning Trees

The right way to count simplicial trees:

$$\tau(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \qquad \text{(unweighted)}$$

$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{\phi \in \Upsilon} x_\phi \qquad \text{(unweighted)}$$

Kalai 1983: $\tau(K_{n_d}) = n^{\binom{n-2}{d}}$ using simplicial Laplacian $\partial\partial^{\mathrm{tr}}$.
(torsion factors arise naturally from Binet-Cauchy expansion)

Subsequent work: Adin 1992 (complete colorful complexes), Petersson, Duval–Klivans–JLM, Lyons, Catanzaro–Chernyak–Klein (all c. 2006–2010)

The Simplicial Matrix-Tree Theorem

Let Δ be a d-dimensional simplicial complex.

Assume $H_k(X;\mathbb{R}) = 0$ for $k = d-1, d-2$.

Let Γ be a $(d-1)$-dimensional spanning tree of Δ.

The **reduced simplicial Laplacian** L_Γ is the square matrix obtained from $\partial_d \partial_d^{tr}$ by deleting the rows and columns corresponding to facets of Γ.

Then,

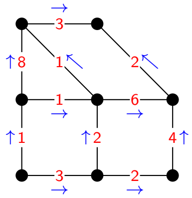
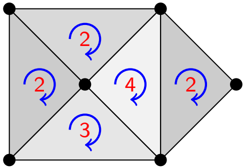
$$\tau(\Delta) = \frac{|H_{d-2}(\Delta;\mathbb{Z})|^2}{|H_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

(In practice, the torsion junk often goes away.)

4. Simplicial Networks and Applications

Simplicial Networks

Simplicial network: pure d-complex with resistances $(r_\phi)_{\phi\in\mathbf{\Phi}}$

$d = 1$   $d = 2$

Currents $\mathbf{i} = (i_\phi)_{\phi\in\mathbf{\Phi}}$ Voltages $\mathbf{v} = (v_\phi)_{\phi\in\mathbf{\Phi}}$

Ohm's law	$i_\phi r_\phi = v_\phi$ for all $\phi \in \Phi$
Kirchhoff's current law	$\mathbf{i} \in \ker(\partial_d)$
Kirchhoff's voltage law	$\mathbf{v} \in \ker(\partial_d)^\perp$

- Dirichlet, Rayleigh, R^{eff} have natural simplicial analogues.
- Attach a unit current generator σ and minimize energy. Then $R^{\text{eff}}_\sigma = v_\sigma$.

Counting Simplicial Trees via Effective Resistance

Theorem [Kook–Lee 2018]
Let $(\Delta, \mathbf{r})$ be a simplicial network and σ a current generator. Then:

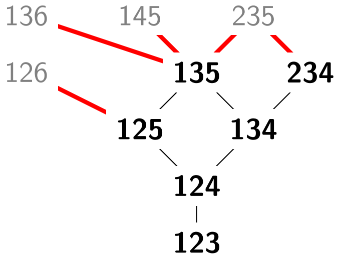
$$R_\sigma^{\text{eff}} = \frac{\hat{\tau}(\Delta/\sigma)}{\hat{\tau}(\Delta)} = \frac{\displaystyle\sum_{T \in \mathcal{T}(\Delta/\sigma)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\phi \in T} r_\phi^{-1}}{\displaystyle\sum_{T \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\phi \in T} r_\phi^{-1}}.$$

- Generalizes Thomassen's theorem for R^{eff} in graphs
- $\Delta/\sigma =$ quotient space (not simplicial, but close enough)
- Application: count trees by induction on facets

Shifted Complexes

A (pure) simplicial complex Δ on vertices $\{1, \ldots, n\}$ is **shifted** if any vertex of a face may be replaced with a smaller vertex.

Equivalently, the facets of Δ form an order ideal in *Gale order* or *componentwise order* (best explained by a picture)



$\Delta = \langle 135, 234 \rangle_{\text{Gale}}$

Facets

Nonfaces

Critical pairs

Shifted complexes are **nice:** shellable, good h-vectors, arise in algebra, Gröbner degenerations of arbitrary complexes. . .

Shifted Complexes

Duval–Klivans–JLM '09: recursion for $\hat{\tau}(\Delta)$ via the shifted complexes $\langle \phi \in \Delta \mid 1 \in \phi \rangle$ and $\langle \phi \in \Delta \mid 1 \notin \phi \rangle$.

Here $\hat{\tau}(\Delta)$ is the finely weighted degree enumerator

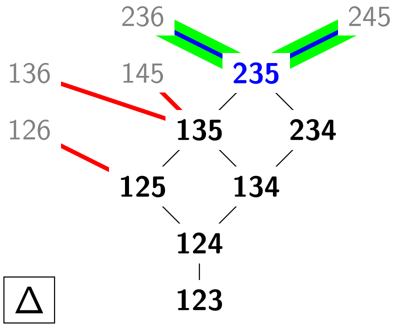
$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon; \mathbb{Z})|^2 \prod_{\substack{\text{facets} \\ \{v_0 < \cdots < v_d\}}} x_{0,v_0} \cdots x_{d,v_d}$$

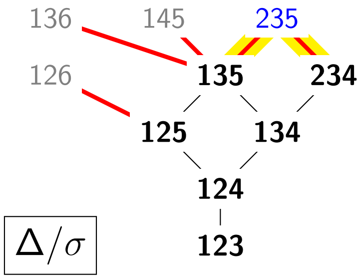
Punchline: Critical pairs P correspond to factors f_P of $\hat{\tau}(\Delta)$.

Duval–Kook–Lee–JLM '21^{+}:

- Calculate $R^{\text{eff}} = \hat{\tau}(\Delta/\sigma)/\hat{\tau}(\Delta)$ for a Gale-maximal face σ
- Show that
$$R^{\text{eff}} = \frac{\prod_{P \text{ vanishes}} f_P}{\prod_{P \text{ appears}} f_P}.$$

Shifted Complexes





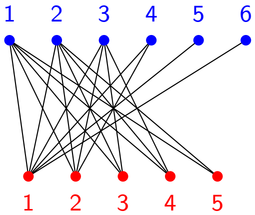
$$R^{\text{eff}}(\sigma) = \frac{\hat{\tau}(\Delta/\sigma)}{\hat{\tau}(\Delta)} = \frac{\prod_{\text{yellow } P} f_P}{\prod_{\text{green } P} f_P}$$

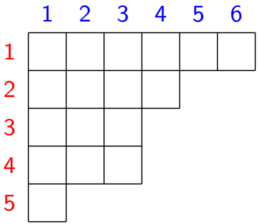
Color-Shifted Complexes

A simplicial complex Δ^d is **color-shifted** [Babson–Novik '06] if:

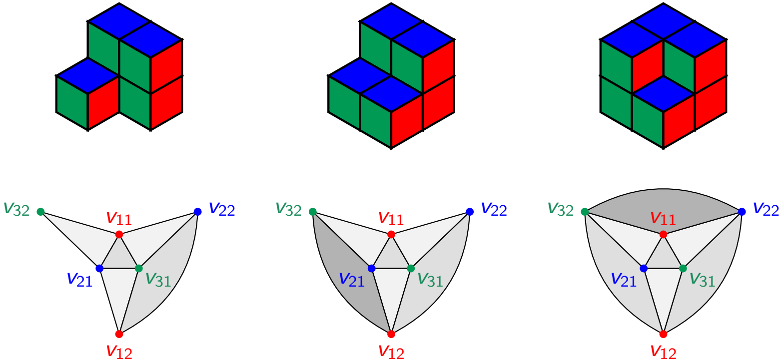
- $V(\Delta) = V_1 \cup \cdots \cup V_{d+1}$, where $V_q = \{v_{q1}, \ldots, v_{q\ell_q}\}$
- Each facet contains exactly one vertex of each color
- A vertex may be replaced with a smaller vertex of same color

A 1-dimensional color-shifted complex is just a Ferrers graph.





Color-Shifted Complexes



Trees in Color-Shifted Complexes

Vertex-weighted spanning tree enumerators:

$$\begin{aligned}\hat{\tau}(\Delta) &= \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{\phi \in \Upsilon} \prod_{v_{qj} \in \phi} x_{qj} \\ &= \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{q,j} x_{qj}^{\deg_\Upsilon(v_{qj})}\end{aligned}$$

Proposition [Duval–Kook–Lee–JLM 2021^+]
Let Δ^d color-shifted, $\sigma = v_{1,k_1} v_{2,k_2} \dots v_{d+1,k_{d+1}} \notin \Delta$.
Then:

$$R^{\mathrm{eff}}(\sigma) = \frac{\hat{\tau}(\Delta + \sigma)}{\tau(\Delta)} = \prod_{q=1}^{d+1} \frac{x_{q,1} + \cdots + x_{q,k_q}}{x_{q,1} + \cdots + x_{q,k_q - 1}}.$$

Trees in Color-Shifted Complexes

Theorem [Duval–Kook–Lee–JLM 2021$^+$]

$$\hat{\tau}(\Delta) = \prod_{q,i} x_{q,i}^{e(q,i)} \prod_{\substack{\rho \in \Delta \\ \dim \rho = d-1}} (x_{m(\rho),1} + \cdots + x_{m(\rho),k(\rho)})$$

where

$$\begin{aligned} e(q,i) &= \#\{\sigma \in \Delta_d \mid v_{q,i} \in \sigma \text{ and } v_{q',1} \in \sigma \text{ for some } q' \neq q\} \\ m(\rho) &= \text{unique color missing from } \rho \\ k(\rho) &= \max\{j \mid \rho \cup v_{m(\rho),j} \in \Delta\} \end{aligned}$$

- Special case $d = 1$ is Ehrenborg–van Willigenburg
- Previously conjectured by Aalipour and Duval [unpublished]
- Result seems inaccessible without effective resistance

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Thank you!