Trees and How to Count Them

Jeremy L. Martin
Department of Mathematics
University of Kansas

The Frank S. Brenneman Lectures
Tabor College
March 28, 2017

Graphs

A **graph** is a pair $G = (V, E)$, where

- V is a set of **vertices**, and
- E is a set of **edges**, each joining two vertices (its **endpoints**).

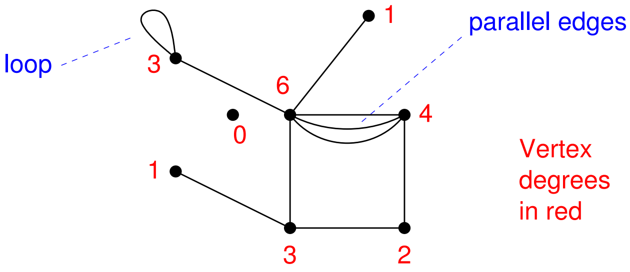
The **degree** of a vertex is the number of edges incident to it.

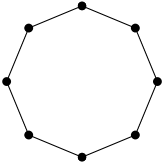
Graphs

A **graph** is a pair $G = (V, E)$, where

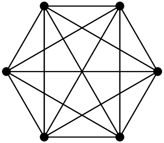
- V is a set of **vertices**, and
- E is a set of **edges**, each joining two vertices (its **endpoints**).

The **degree** of a vertex is the number of edges incident to it.

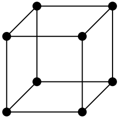




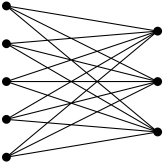
Cycle graph C_8



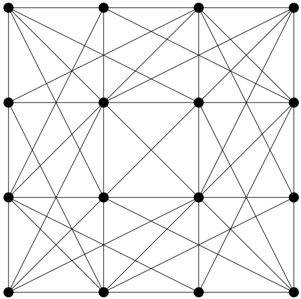
Complete graph K_6



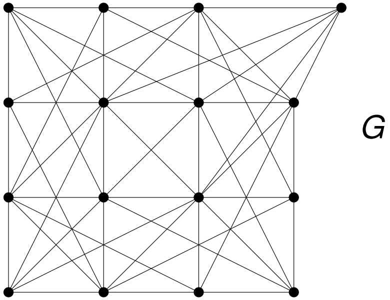
Cube graph Q_3



Complete bipartite graph $K_{5,3}$



G


G

Why study graphs?

- Real-world applications
  - Combinatorial optimization (routing, scheduling. . .)
  - Computer science (data structures, sorting, searching. . .)
  - Biology (evolutionary descent. . .)
  - Chemistry (molecular structure. . .)
  - Engineering (roads, rigidity. . .)
  - Network models (social networks, the Internet. . .)
- Pure mathematics
  - Combinatorics (ubiquitous!)
  - Discrete dynamical systems (chip-firing game. . .)
  - Algebra (quivers, Cayley graphs. . .)
  - Discrete geometry (polytopes, sphere packing. . .)

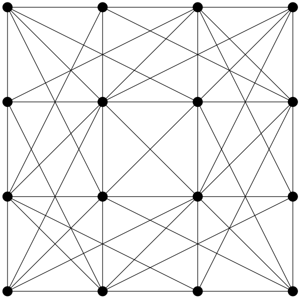
Spanning Trees

Definition A **spanning tree** of a graph G is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

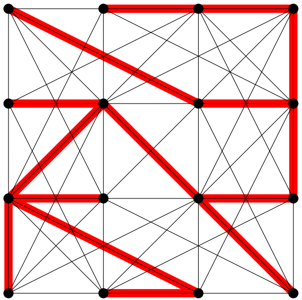
Any two of these conditions together imply the third.

Spanning Trees



G

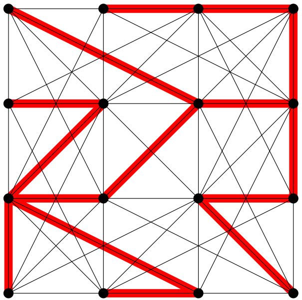
Spanning Trees



G

T_1

Spanning Trees



G

T_2

Counting Spanning Trees

$\mathscr{T}(G)$ = set of spanning trees of G
$\tau(G)$ = number of spanning trees of G

- $\tau(\text{tree}) = 1$
- $\tau(C_n) = n$
- $\tau(K_n) = n^{n-2}$ (Cayley's formula; highly nontrivial!)
- $\tau(K_{m,n}) = n^{m-1}m^{n-1}$
- Many other enumeration formulas for nice graphs

Deletion and Contraction

Let $e \in E(G)$.

Deletion and Contraction

Let $e \in E(G)$.

- **Deletion** $G - e$: Remove e

Deletion and Contraction

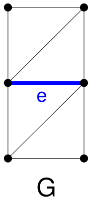
Let $e \in E(G)$.

- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point

Deletion and Contraction

Let $e \in E(G)$.

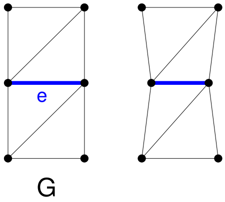
- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



Deletion and Contraction

Let $e \in E(G)$.

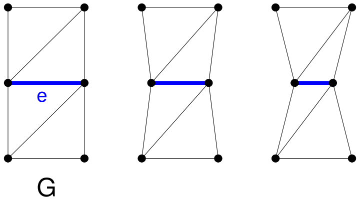
- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



Deletion and Contraction

Let $e \in E(G)$.

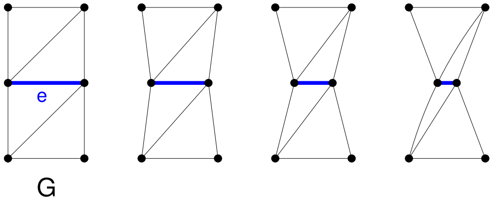
- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



Deletion and Contraction

Let $e \in E(G)$.

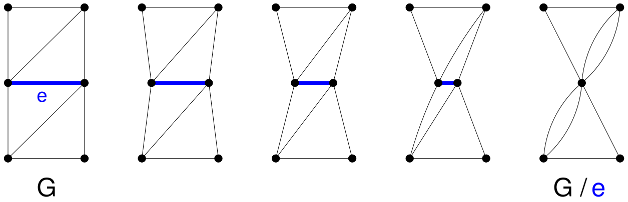
- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



Deletion and Contraction

Let $e \in E(G)$.

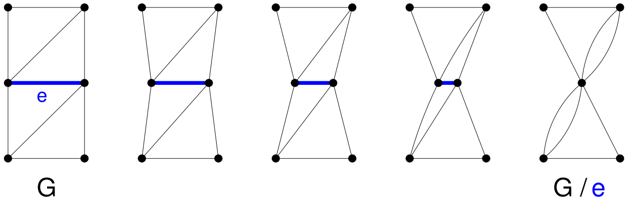
- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



Deletion and Contraction

Let $e \in E(G)$.

- **Deletion** $G - e$: Remove e
- **Contraction** G/e: Shrink e to a point



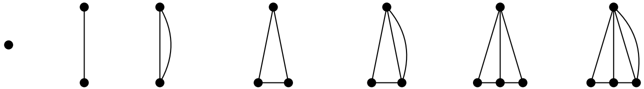
Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$.

Deletion and Contraction

Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$.

This formula allows easy calculation of $\tau(G)$ and some fun results:

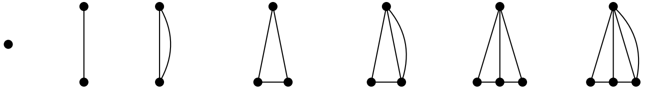
G



Deletion and Contraction

Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$.

This formula allows easy calculation of $\tau(G)$ and some fun results:



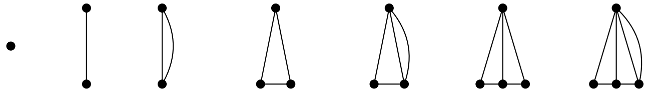
G							
$\tau(G)$	1	1	2	3	5	8	13

Deletion and Contraction

Theorem $\tau(G) = \tau(G - e) + \tau(G/e)$.

This formula allows easy calculation of $\tau(G)$ and some fun results:

G



$\tau(G)$: 1, 1, 2, 3, 5, 8, 13

Unfortunately:

- "easy" does not mean "efficient": $2^{|E|}$ steps are required to calculate $\tau(G)$ this way.
- Useful only for graph families with recursive deletion/contraction structure (*not* K_n, $K_{m,n}$, Q_n, etc.).

The Matrix-Tree Theorem

Definition Let G be a connected graph with vertices $1, \ldots, n$ and no loops. The **Laplacian** of G is the $n \times n$ matrix $L = [\ell_{ij}]$:

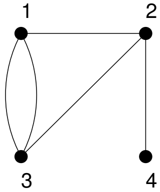
$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -(\text{number of edges from } i \text{ to } j) & \text{if } i \neq j. \end{cases}$$

- L is symmetric and positive semi-definite
  - $L = \partial\partial^T$, where $\partial =$ signed vertex-edge incidence matrix
- rank $L = n - 1$
- ker L is spanned by the all-1's vector

The Matrix-Tree Theorem

Example

$G =$



$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} \ .$$

The Matrix-Tree Theorem

The Matrix-Tree Theorem (Kirchhoff, 1847)

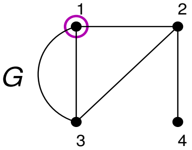
(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} \ .$$

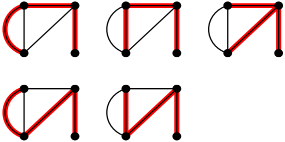
(2) Let $1 \leq i \leq n$. Form the **reduced Laplacian** L_i by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i \ .$$

The Matrix-Tree Theorem: Example



$\tau(G) = 5$



$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

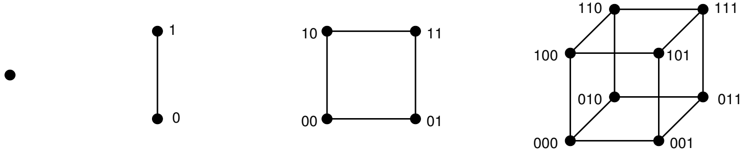
Eigenvalues: 0, 1, 4, 5

$(1 \cdot 4 \cdot 5)/4 = 5$

$\det L_1 = 5$

Hypercubes

The **hypercube graph** Q_n has 2^n vertices, labeled by strings of n bits (0's and 1's), with two vertices adjacent if they agree in all but one bit.



Q_0 Q_1 Q_2 Q_3

Theorem The eigenvalues of the Laplacian of Q_n are $0, 2, 4, \ldots, 2n$, with $2k$ having multiplicity $\binom{n}{k}$. Therefore,

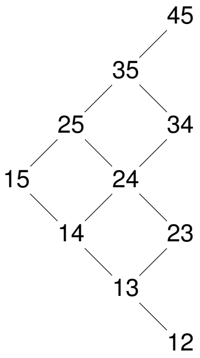
$$\tau(Q_n) = 2^{2^n - n - 1} \prod_{k=2}^{n} k^{\binom{n}{k}}.$$

Threshold Graphs

A graph with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \leq a$ and $b' \leq b$.

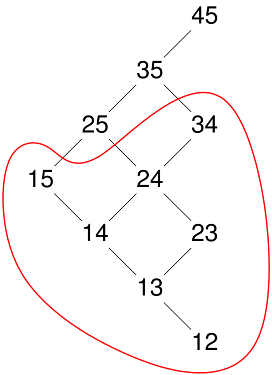
Threshold Graphs

A graph with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \le a$ and $b' \le b$.



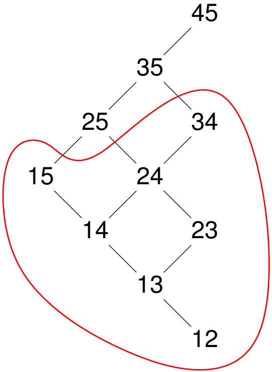
Threshold Graphs

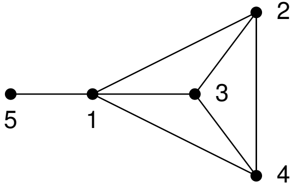
A graph with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \leq a$ and $b' \leq b$.



Threshold Graphs

A graph with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \leq a$ and $b' \leq b$.





Threshold Graphs

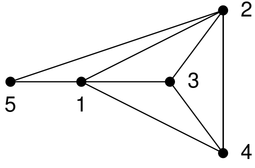
Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $1, \ldots, n$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.

Threshold Graphs

Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $1, \ldots, n$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

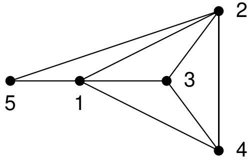
Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.

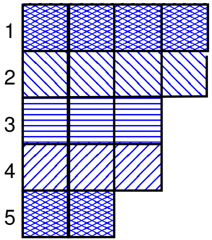


Threshold Graphs

Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $1, \ldots, n$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.



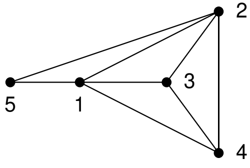


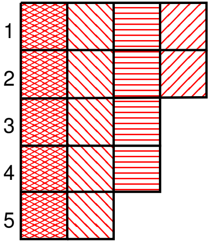
Vertex degrees: 4, 4, 3, 3, 2

Threshold Graphs

Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $1, \dots, n$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.



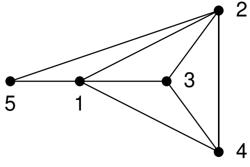


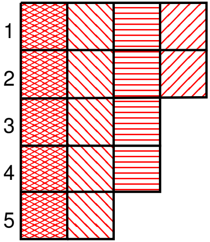
Laplacian eigenvalues: 5, 5, 4, 2, 0

Threshold Graphs

Theorem [Merris 1994] The eigenvalues of the Laplacian of a threshold graph G on vertices $1, \ldots, n$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.





$\tau = 5 \times 4 \times 2 = 40$

Laplacian eigenvalues: 5, 5, 4, 2, 0

Weighted Counting

Theorem [Cayley–Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_1^{\deg_T(1)} \cdots x_n^{\deg_T(n)} \;=\; x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}$$

- Setting $x_i = 1$ for all i recovers $\tau(K_n) = n^{n-2}$
- Can be proved either bijectively (*Prüfer code*) or by a souped-up version of the Matrix-Tree Theorem
- Other weighted tree counting formulas:
  - *Via bijections*: Fiedler-Sedláček (complete bipartite graphs), Knuth, Kelmans, Remmel-Williamson, etc.
  - *Via MTT*: JLM–Reiner (threshold graphs, hypercubes)

Weighted Tree Counts for Threshold Graphs

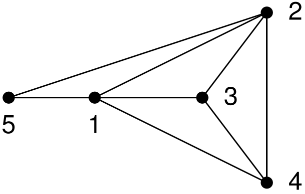
Theorem [JLM–Reiner 2005] Let G be a threshold graph on vertices $1, \ldots, n$ with degree sequence λ. Weight each edge $e = ij$ with $i < j$ by $x_i y_j$. Then the bidegree generating function is

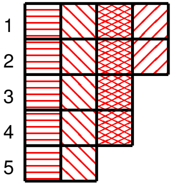
$$\sum_{T \in \mathscr{T}(G)} \prod_{e: i<j} x_i y_j = x_1 y_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_{\min(i,r)} y_{\max(i,r)} \right)$$

and therefore (setting $y_i = x_i$) the degree generating function is

$$\sum_{T \in \mathscr{T}(G)} \prod_{i=1}^{n} x_i^{\deg(i)} = x_1 \cdots x_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_i \right)$$

Weighted Tree Counts for Threshold Graphs





Bidegree generating function:

$$x_1\, y_5\, (x_1 y_2 + x_2 y_2 + x_2 y_3 + x_2 y_4 + x_2 y_5) \\ \times \quad (x_1 y_3 + x_2 y_3 + x_3 y_3 + x_3 y_4)(x_1 y_4 + x_2 y_4)$$

Degree generating function:

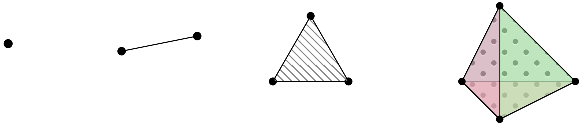
$$x_1\, x_2\, x_3\, x_4\, x_5\, (x_1 + x_2 + x_3 + x_4 + x_5)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2)$$

Simplicial Complexes

A **d-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.

Simplicial Complexes

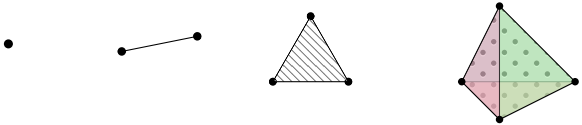
A **d-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.



$d = 0$ $d = 1$ $d = 2$ $d = 3$

Simplicial Complexes

A **d-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.

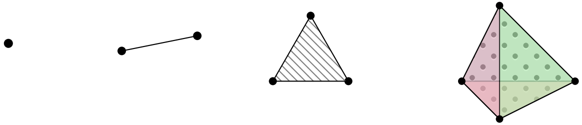


$d=0$ $d=1$ $d=2$ $d=3$

A **simplicial complex** is a space built (properly!) from simplices.

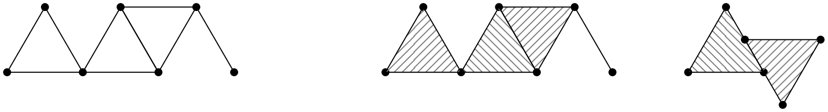
Simplicial Complexes

A **d-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.



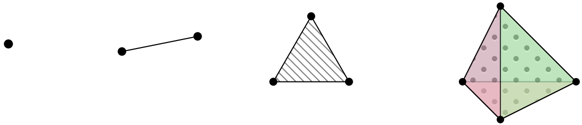
$d=0$ $d=1$ $d=2$ $d=3$

A **simplicial complex** is a space built (properly!) from simplices.



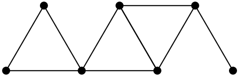
Simplicial Complexes

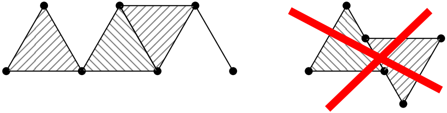
A **d-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.



$d = 0$ $d = 1$ $d = 2$ $d = 3$

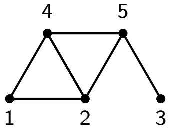
A **simplicial complex** is a space built (properly!) from simplices.



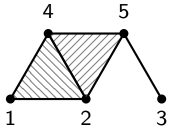


Simplicial Complexes

Combinatorially, a simplicial complex is a set family $\Delta \subseteq 2^{\{1,2,\dots,n\}}$ such that if $\sigma \in \Delta$ and $\sigma' \subseteq \sigma$, then $\sigma' \in \Delta$.



$\Delta_1 = \langle 12, 14, 24, 24, 25, 35 \rangle$



$\Delta_2 = \langle 124, 245, 35 \rangle$

- **faces** or **simplices**: elements of Δ
- **dimension:** $\dim \sigma = |\sigma| - 1$
- **facet**: a maximal face
- **pure** complex: all facets have equal dimension

Simplicial Spanning Trees

Definition Let Δ be a simplicial complex of dimension d.

A subcomplex $\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** (SST) if:

1. Υ contains all simplices of Δ of dimension $< d$.
2. Υ is "acyclic" and "connected".
   - *Technically:* $\tilde{H}_d(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$.
   - *Intuitively:* Υ has no "bubbles" whose boundary is an orientable d- or $(d-1)$-manifold.

As before, we'll write $\mathscr{T}(\Delta)$ for the set of SSTs of Δ.

Examples of SSTs

- $\dim \Delta = 1$: $\mathscr{T}(\Delta) =$ graph-theoretic spanning trees

Examples of SSTs

- $\dim \Delta = 1$: $\mathscr{T}(\Delta) =$ graph-theoretic spanning trees
- $\dim \Delta = 0$: $\mathscr{T}(\Delta) =$ vertices of Δ

Examples of SSTs

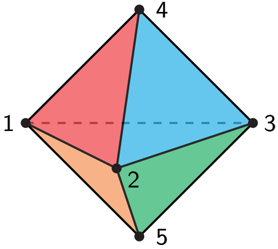
- $\dim \Delta = 1$: $\mathscr{T}(\Delta) =$ graph-theoretic spanning trees
- $\dim \Delta = 0$: $\mathscr{T}(\Delta) =$ vertices of Δ
- If Δ is contractible: it has only one SST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$

Examples of SSTs

- $\dim \Delta = 1$: $\mathscr{T}(\Delta) =$ graph-theoretic spanning trees
- $\dim \Delta = 0$: $\mathscr{T}(\Delta) =$ vertices of Δ
- If Δ is contractible: it has only one SST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- If Δ is a simplicial sphere: SSTs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any facet (maximal face)
  - Simplicial spheres are analogous to cycle graphs

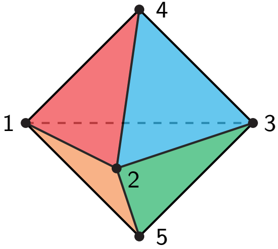
Examples of SSTs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



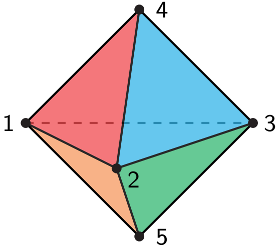
Examples of SSTs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



Solution: 15.

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?

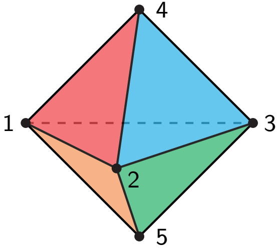


Solution: 15.

- Either remove triangle 123 and any other triangle (6 SSTs)...

Examples of SSTs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?

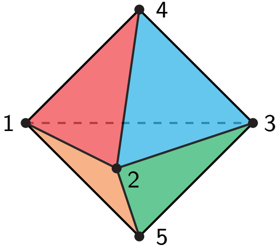


Solution: 15.

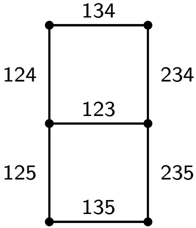
- Either remove triangle 123 and any other triangle (6 SSTs)...
- ...or one each "northern" and "southern" triangle (9 SSTs).

Examples of SSTs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



Solution: 15.

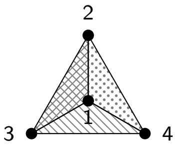


- Either remove triangle 123 and any other triangle (6 SSTs)...
- ...or one each "northern" and "southern" triangle (9 SSTs).

Counting SSTs: The Good News

Every simplicial complex Δ of dimension d has a **Laplacian matrix** $L = L(\Delta) = [\ell_{ij}]$ with

- rows and columns indexed by $(d-1)$-faces (**ridges**) of Δ
- ℓ_{ii} = number of facets containing i
- $\ell_{ij} = \pm 1$ if ridges i, j lie in a common facet, 0 otherwise



Δ

$$\begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$L(\Delta)$

Counting SSTs: The Good News

Simplicial Matrix-Tree Theorem
(Bolker, Kalai, Adin, Duval–Klivans–JLM, . . .)

Let Δ be a simplicial complex of dimension d.

Form a **reduced Laplacian** $L_T(\Delta)$ from $L(\Delta)$ by deleting the rows and columns corresponding to a $(d-1)$-dimensional SST $T \subseteq \Delta$.

Then the "number" of spanning trees of Δ is $\det L_T$, divided by a correction factor given by T.

Counting SSTs: The Bad News

In dimension $d \geq 2$, spanning trees can have torsion.

- *Technically:* Torsion of $\Upsilon \in \mathscr{T}(\Delta) = |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|$
- *Intuitively:* Some piece of Υ is twisted in a funny way (e.g., a non-orientable d–manifold)

Simplicial Matrix-Tree Theorem

$$\tau(\Delta) \overset{\text{def}}{=} \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = \text{(correction factor)} \times \det \hat{L}_T$$

- If $d = 1$ then all summands are 1!
- In many natural cases, the correction factor is 1 as well.

Kalai's Theorem

Simplicial generalization of the complete graph:

$$K_{n,d} = \{F \subseteq \{1, \ldots, n\} \mid \dim F \le d\}$$

Kalai's Theorem

Simplicial generalization of the complete graph:

$$K_{n,d} = \{F \subseteq \{1,\dots,n\} \mid \dim F \leq d\}$$

Theorem [Kalai 1983]

$$\tau(K_{n,d}) \ = \ n^{\binom{n-2}{d}}.$$

More generally,

$$\sum_{\Upsilon \in \mathscr{T}(K)} |\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{i=1}^{n} x_i^{\deg_\Upsilon(i)} = (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}.$$

Kalai's Theorem

- Kalai's theorem reduces to $\tau(K_n) = n^{n-2}$ when $d = 1$, and the weighted version reduces to Cayley-Prüfer.
- Bolker (1976): Observed that $n^{\binom{n-2}{d}}$ is an exact count of trees for small n, d, but fails for $n = 6$, $d = 2$.
  - The problem is torsion — $\mathbb{RP}^2$ requires six vertices to triangulate
- Adin (1992): Analogous formula for **complete colorful complexes**, generalizing $\tau(K_{n,m}) = n^{m-1}m^{n-1}$

Shifted Simplicial Complexes

A simplicial complex Δ with vertex set $\{1, 2, \ldots, n\}$ is **shifted** if whenever $a_1 a_2 \cdots a_k \in \Delta$ and $b_i \leq a_i$ for all i, then $b_1 b_2 \cdots b_k \in \Delta$.

(So one-dimensional shifted complexes are just threshold graphs.)

Theorem [Duval–Reiner 2002]

Let $\lambda_i =$ number of max-dim faces containing i.
Then eigenvalues of $L(\Delta) =$ column lengths of λ.
(Generalization of Merris' Theorem)

Theorem [Duval–Klivans–JLM 2009]

Factorization of multidegree g.f. for spanning trees of a shifted complex. (Generalization of JLM–Reiner formula)

Further Directions

- Theory of SSTs and the Matrix-Tree Theorem generalize easily from simplicial complexes to cell complexes
  - Cubes and their skeletons [Duval–Klivans–JLM 2011], [Aalipour–Duval–Kook–Lee–JLM 2017$^+$]
  - Cellular MTT discovered independently in contexts of probability [Lyons 2009] and mathematical physics [Catanzaro–Chernyak–Klein 2015]
- Simplicial/cell complexes that have integer Laplacian eigenvalues "should" have factorizable weighted tree g.f.'s
  - Matroid complexes; others?
- Critical groups:
  - Complex $\Delta \Rightarrow$ abelian group $K(\Delta)$ of size $\tau(\Delta)$
  - Cuts, flows, sandpile theory, "algebraic geometry on graphs"
  - Group structure very mysterious