The **co**critical group of a cell complex

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Cell complexes and combinatorial Laplacians

Throughout, X^d is a finite cell (CW) complex of dimension d.

Acyclization[1] of X: $(d+1)$-dimensional complex Ω such that $\tilde{H}_{d+1}(\Omega;\mathbb{Q}) = \tilde{H}_d(\Omega;\mathbb{Q}) = 0$ and $X = d$-skeleton of Ω

Augmented cellular chain complex of Ω (over $\mathbb{Z}$):

$$\cdots \rightleftarrows C_{i+1} \underset{\partial_{i+1}^*}{\overset{\partial_{i+1}}{\rightleftarrows}} C_i \underset{\partial_i^*}{\overset{\partial_i}{\rightleftarrows}} C_{i-1} \rightleftarrows \cdots$$

(identifying each i-cell with its characteristic function in C^i).

Combinatorial Laplacians (updown and downup):

$$L_i^{\mathrm{ud}} = \partial_i \partial_i^* : C_{i-1} \to C_{i-1} \qquad L_i^{\mathrm{du}} = \partial_{i+1}^* \partial_{i+1} : C_{i+1} \to C_{i+1}$$

[1] Not every complex has an acyclization, but many interesting ones do.

Critical and cocritical groups

Notation: $\mathbf{T}(G)$ = torsion summand of a f.g. abelian group G.

Critical groups of X:

$$K_{i-1}(X) \ := \ \mathbf{T}(\operatorname{coker} L_i^{\mathrm{ud}} : \ C_{i-1} \to C_{i-1})$$

Cocritical groups of X:

$$K_{i+1}^*(X) \ := \ \mathbf{T}(\operatorname{coker} L_{i+1}^{\mathrm{du}} : \ C_{i+1} \to C_{i+1})$$

- Shorthand: $K(X) = K^{d-1}(X)$ and $K^*(X) = K_{d+1}^*(X)$
- $K_{i+1}(X)$ is independent of the choice of acyclization Ω.
- To compute K and K^*, find Smith normal forms of Laplacians.
- X connected graph $\implies$ $K(X)$ = usual critical group (cardinality = number of spanning trees).

Critical groups and cut and flow lattices

Let $n =$ number of i-cells, so $C_i(X, \mathbb{Z}) \cong \mathbb{Z}^n$.

Cut lattice: $\mathcal{C}_i = \operatorname{Im} \partial_i^* \subseteq \mathbb{Z}^n$
Flow lattice: $\mathcal{F}_i = \ker \partial_i \subseteq \mathbb{Z}^n$

Dual of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$:

$$\mathcal{L}^\sharp := \{v \in \mathcal{L} \otimes \mathbb{R}^n \colon \langle v, w \rangle \in \mathbb{Z} \ \ \forall w \in \mathcal{L}\} \cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Z}).$$

Theorem (DKM 12)

$K(X) \cong \mathcal{C}^\sharp/\mathcal{C}$ and $K^*(X) \cong \mathcal{F}^\sharp/\mathcal{F}$.
Moreover, there are short exact sequences

$$0 \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K(X) \to \mathbf{T}(\tilde{H}_{d-1}(X;\, \mathbb{Z})) \to 0,$$
$$0 \to \mathbf{T}(\tilde{H}_{d-1}(X;\, \mathbb{Z})) \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K^*(X) \to 0.$$

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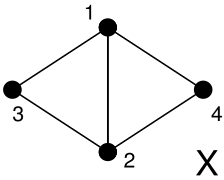
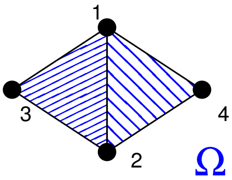
- If $\tilde{H}_{d-1}(X;\, \mathbb{Z})$ is torsion-free (for example, if X is a graph) then $K(X) \cong K^*(X)$.
- Graph case (and motivation for present work): Bacher–de La Harpe–Nagnibeda 1997
- "Torsion makes $K(X)$ bigger and $K^*(X)$ smaller."

Example 1



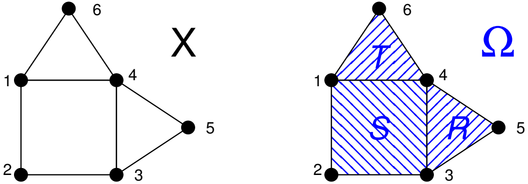


$$\partial_2(\Omega) = \begin{array}{c} \\ 12 \\ 13 \\ 23 \\ 14 \\ 24 \end{array} \begin{array}{c} \begin{array}{cc} 123 & 124 \end{array} \\ \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$L_2^{\mathrm{du}} = \partial_2^* \partial_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

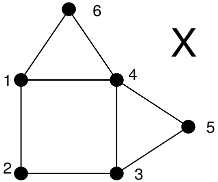
Cokernel: $\mathbb{Z}/8\mathbb{Z} \cong K(X)$

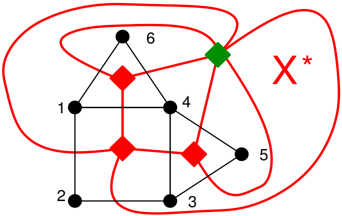
Example 2



$$L_2^{\mathrm{du}}(\Omega) = \begin{array}{c} \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{pmatrix} \end{array}$$

Example 2 and Planar Duality





$$L^{\mathrm{du}}(\Omega) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{pmatrix} = \text{reduced Laplacian of planar dual } X^*$$

Corollary [Cori–Rossin 2000]: If X is a planar graph and X^* is any planar dual then $K(X) \cong K^*(X) \cong K(X^*)$.

Enumerating Cellular Spanning Trees

Recall that when X is a connected graph, $|K(X)|$ = number of spanning trees. More generally

$$|K(X)| = \tau_d(X) := \sum_{\Upsilon} |\mathbf{T}(\tilde{H}_{d-1}(\Upsilon;\, \mathbb{Z}))|^2$$

where Υ ranges over all **cellular spanning forests in** X: subcomplexes with complete $(d-1)$-skeleton such that

- $\tilde{H}_d(\Upsilon;\, \mathbb{Z}) = 0$ ("acyclic") and
- $|\tilde{H}_{d-1}(\Upsilon;\, \mathbb{Q}) = \tilde{H}_{d-1}(X;\, \mathbb{Q})$ ("connected").

(Lyons, DKM, Catanzaro–Chernyak–Klein)

Enumerating Cellular Spanning Trees

Theorem (Lyons 09, DKM 11, Catanzaro–Chernyak–Klein 12)

The critical group counts forests by torsion homology:

$$|K(X)| = \tau_d(X) := \sum_{\textit{forests } \Upsilon \subseteq X} |\mathbf{T}(\tilde{H}_{d-1}(\Upsilon;\ \mathbb{Z}))|^2$$

Theorem (DKM 12)

The cocritical group counts forests by relative torsion homology:

$$|K^*(X)| = \tau_d^*(X) := \sum_{\textit{forests } \Upsilon \subseteq X} |\tilde{H}_d(X, \Upsilon;\ \mathbb{Z})|^2$$

Cellular Spheres

Theorem (DKM 11)

Let X be a cellular sphere with n facets (e.g., the boundary of a convex polytope). Then $K(X) \cong \mathbb{Z}/n\mathbb{Z}$.

Our original proof: Blah blah blah.

New proof: $K(X) \cong K^*(X)$ (since $\tilde{H}_{d-1}(X; \mathbb{Z}) = 0$). Form an acyclization Ω by attaching one $(d+1)$-cell whose boundary is a signed sum of the d-cells. Therefore

$$K^*(X) \cong \operatorname{coker} L^{\mathrm{du}}_{d+1}(\Omega) = \operatorname{coker} \left[n \right] = \mathbb{Z}/n\mathbb{Z}.$$

More Applications

Question: Are there other complexes for which it is easier to compute the cocritical group than the critical group, or at least to count spanning trees?

More Applications

Example 1: X = octahedron subdivided into eight tetrahedra; $f(X) = (1, 7, 18, 20, 8)$.

How many spanning 2-trees does X have?

- $L_1^{\text{ud}}(X) = \partial_2 \partial_2^* =$ some 18×18 matrix
- $L_3^{\text{du}}(X) = \partial_3^* \partial_3 = I + L(Q_3)$ (Q_3 = cube graph)
- Eigenvalues of $L(Q_3)$: $0, 2, 2, 2, 4, 4, 4, 6$
- Eigenvalues of $I + L(Q_3)$: $1, 3, 3, 3, 5, 5, 5, 7$

$$\boxed{\boldsymbol{\tau_2(X) = 3^3 \cdot 5^3 \cdot 7.}}$$

(Note: L_1^{ud} has integer eigenvalues.)

More Applications

Example 1: X = octahedron subdivided into eight tetrahedra

Example 2: Y = polyhedral cell complex from X obtained by "puffing up" each tetrahedron into a bipyramid.

- $L_3^{\mathrm{du}}(Y) = \partial_3^* \partial_3 = 3I + L(Q_3)$
- Eigenvalues of $L(Q_3)$: $0, 2, 2, 2, 4, 4, 4, 6$
- Eigenvalues of $3I + L(Q_3)$: $3, 5, 5, 5, 7, 7, 7, 9$

$$\boxed{\tau_2(\mathbf{Y}) = 3 \cdot 5^3 \cdot 7^3 \cdot 9.}$$

- $L_1^{\mathrm{ud}}(Y)$ does **not** have integer eigenvalues.

Some Questions

1. For some small complexes, L^{ud}_{i-1} and L^{du}_{i+1} are simultaneously Laplacian integral. Is this a coincidence or is there some connection between their spectra?
2. Are there (families of) complexes other than spheres for which the structure of $K^*(X)$ can easily be determined?
3. Generalization: (co)critical groups of arbitrary chain complexes — it is still the case that $K_{i-1} = K^*_{i+1}$ if there is no torsion homology

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Thanks for listening!