Chromatic Symmetric Functions and Polynomial Invariants of Trees

José Aliste-Prieto[1] Jeremy L. Martin[2]
Jennifer D. Wagner[3] José Zamora[1]

[1]Universidad Andres Bello, Chile
[2]University of Kansas, USA
[3]Washburn University, USA

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Chromatic Symmetric Functions of Graphs

Let $G = (V, E)$ be a simple graph with $V = [n] = \{1, \ldots, n\}$.

proper coloring: $f : V \to \mathbb{N}_{>0}$ with $f(i) \neq f(j)$ whenever $ij \in E$

chromatic symmetric function (CSF): the power series

$$\mathbf{X}_G = \mathbf{X}_G(x_1, x_2, \ldots) = \sum_{\substack{f : V \to \mathbb{N}_{>0} \\ \text{proper}}} x_{f(1)} \cdots x_{f(n)}.$$

- Symmetric and homogeneous of degree n
- Generalizes the chromatic polynomial:

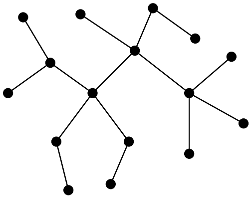
$$\mathbf{X}_G(1^k, 0^\infty) = \text{ number of proper } k\text{-colorings}$$

Chromatic Symmetric Functions of Graphs

- Introduced by Stanley in 1995
- **Related invariants:** Tutte symmetric function / U-polynomial (Noble–Welsh 1999), matroid quasisymmetric function (Billera–Jia–Reiner 2009)
- **Analogues:** noncommutative CSFs (Gebhard–Sagan 2001), quasisymmetric CSFs (Shareshian–Wachs 2016), ...
- **Applications:** combinatorial Hopf algebras (Aguiar–Bergeron–Sottile 2006), cohomology of Hessenberg subvarieties of flag manifolds (Shareshian–Wachs 2012)

Describing CSFs Finitely

Let $G = (V, E)$ be a graph, $n = |V|$, $A \subseteq E$
type of A = partition of n whose parts are component sizes of $G|_A$



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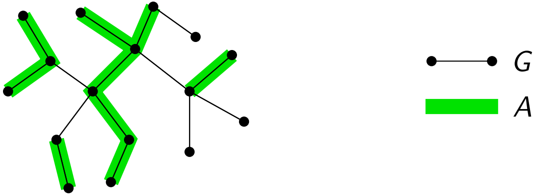

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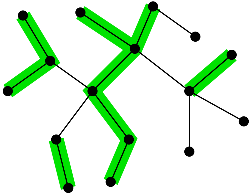

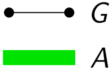

$\text{type}(A) = (6, 3, 2, 2, 1, 1, 1)$

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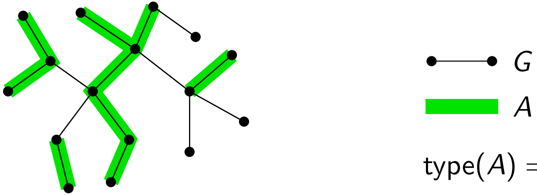


$\text{type}(A) = (6, 3, 2, 2, 1, 1, 1)$

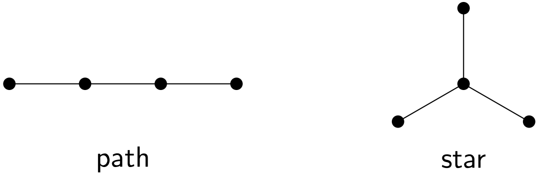
Theorem (Stanley 1995 / JLM–Morin–Wagner 2008)
The CSF of a tree T is determined by the **type numbers**

$$c_\lambda(T) = \#\{A \subseteq E \mid \text{type}(A) = \lambda\}$$

for all partitions λ of n.

Distinguishing Trees with the CSF

Example: The two trees with $n = 4$ are the path and the star.



Type numbers $c_\lambda(T)$:

λ	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
path	1	3	1	2	1
star	1	3	0	3	1

In fact $\mathbf{X}_T \neq \mathbf{X}_{T'}$ for all non-isomorphic trees T, T' with $n \leq 29$ [Heil–Ji 2019]

Question (Stanley)

Is a tree uniquely determined up to isomorphism by its CSF?

I.e., if T, T' are non-isomorphic trees, must $X(T) \neq X(T')$?

Or, stated more broadly:

Can the local structure of a tree be recovered from global data?

Stanley's Uniqueness Problem

Question (Stanley)
Is a tree uniquely determined up to isomorphism by its CSF?

I.e., if T, T' are non-isomorphic trees, must $X(T) \neq X(T')$?

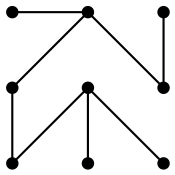
Or, stated more broadly:
Can the local structure of a tree be recovered from global data?

- There are infinitely many pairs of non-acyclic graphs with the same CSFs [Orellana–Scott 2016].
- All trees on n vertices have the same chromatic *polynomial*; the CSF is much stronger.

Numerical Tree Invariants

Numerical data sets associated with a tree $T = (V, E)$:

- type numbers (from CSF)



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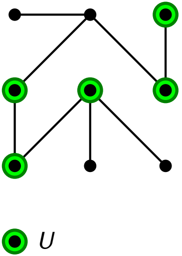
- vertex sets $U \subseteq V$ with a vertices, b external edges, and c internal edges



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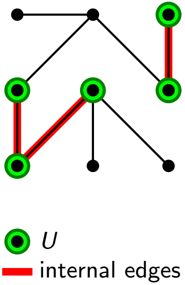

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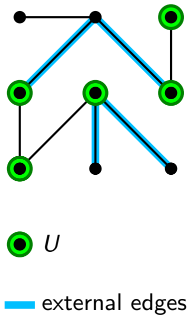

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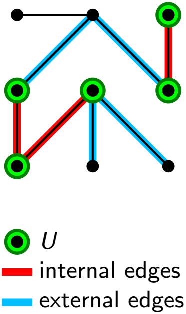

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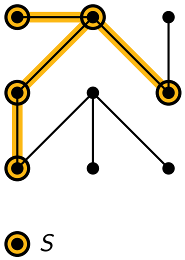
Numerical data sets associated with a tree $T = (V, E)$:

- type numbers (from CSF)
- vertex sets $U \subseteq V$ with a vertices, b external edges, and c internal edges
- subtrees S with q edges and r external edges
- subtrees S with q edges and s leaves



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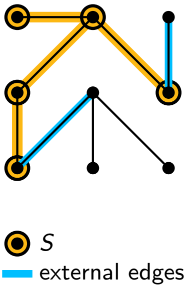

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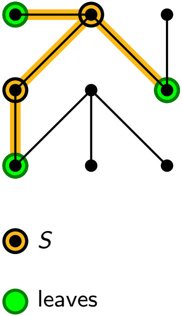

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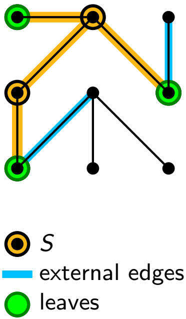

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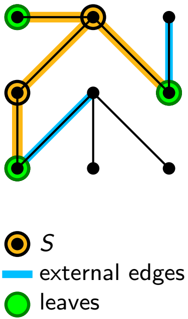
- vertices of degree d



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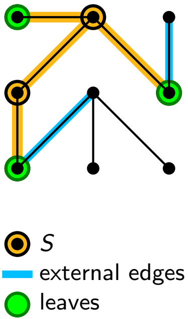
- vertices of degree d
- paths of length ℓ



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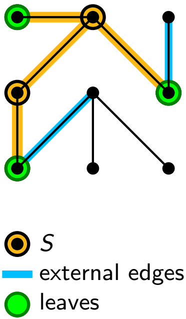


Which of these data sets determine the others?

Relationships Between Tree Invariants

chromatic symmetric function/ type numbers

vertex sets (size, internal edges, external edges)

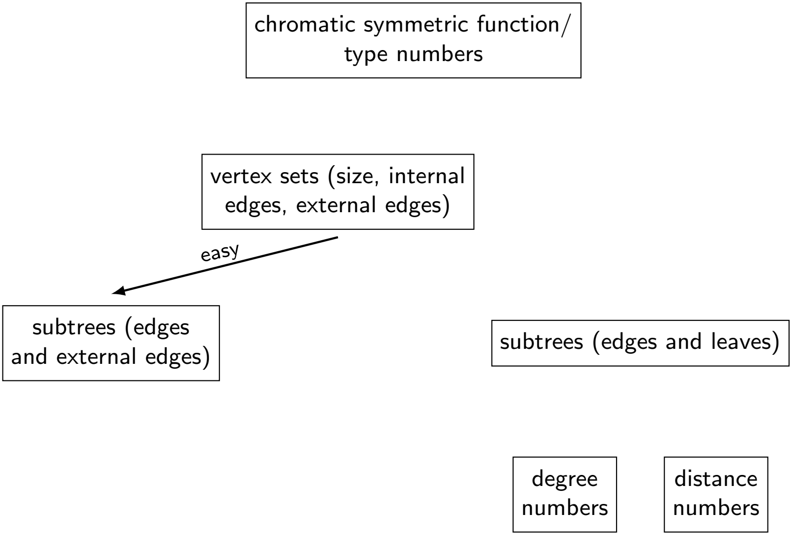
subtrees (edges and external edges)

subtrees (edges and leaves)

degree numbers

distance numbers

Relationships Between Tree Invariants



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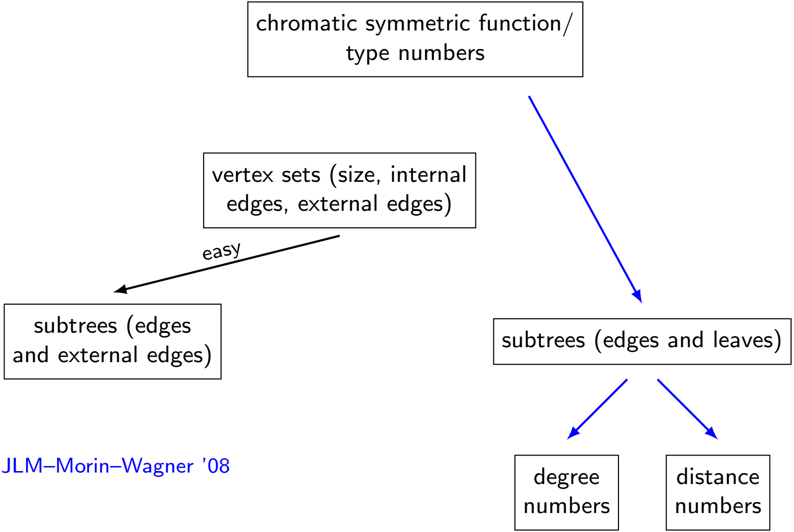

JLM–Morin–Wagner '08

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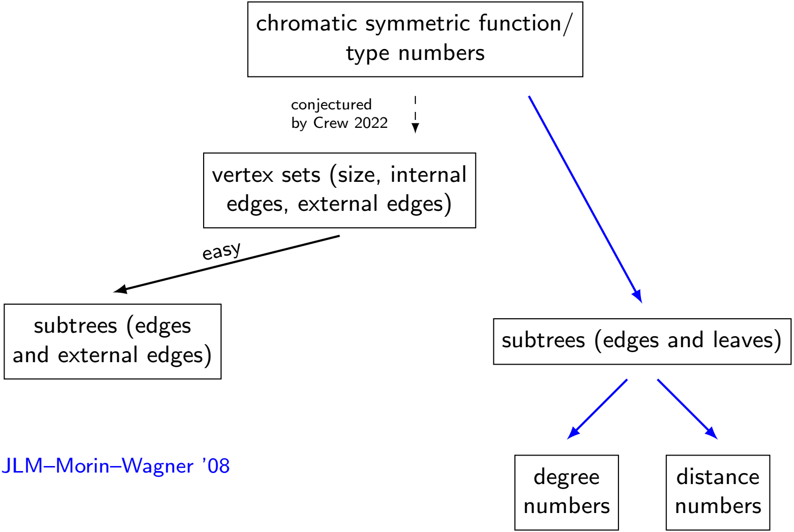

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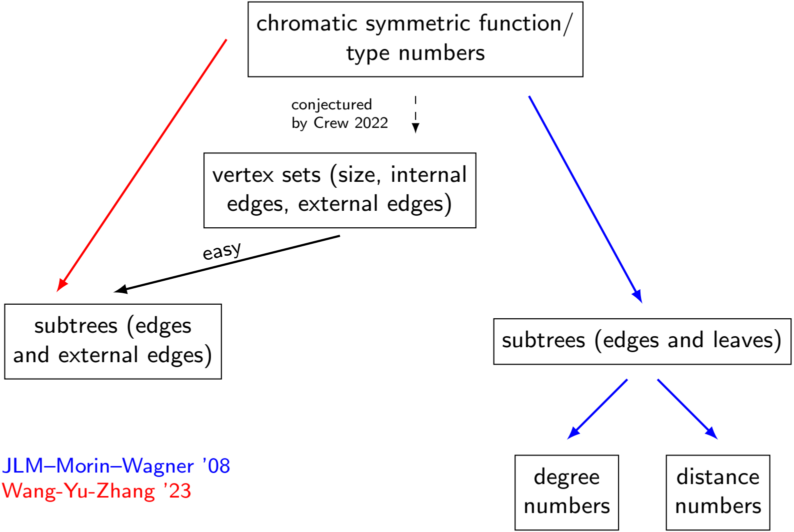

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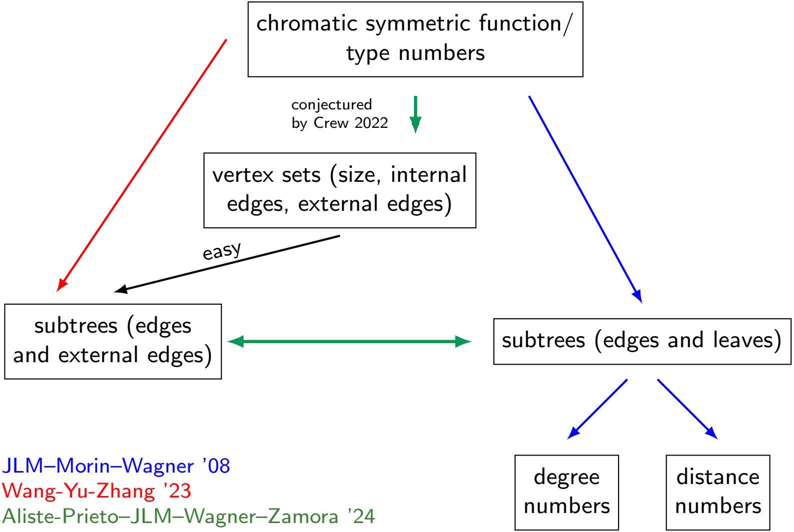


JLM–Morin–Wagner '08
Wang-Yu-Zhang '23
Aliste-Prieto–JLM–Wagner–Zamora '24

Crew's Conjecture: Obtaining the GDP from the CSF

Theorem [Aliste-Prieto, JLM, Wagner, Zamora 2024]

The CSF determines the vertex set data *linearly*.
That is, for every tree T, the numbers

$$f_T(a,b,c) := \#\{A \subseteq V(T) \colon\ |A| = a,\ d(A) = b,\ e(A) = c\}$$

(where $d(A)$ and $e(A)$ are the numbers of external and internal edges) are given by the formula

$$f_T(a,b,c) = \sum_{\lambda \vdash n} c_\lambda(T)\omega(\lambda,a,b,c)$$

where $\omega(\lambda,a,b,c)$ are integers defined independently of T.

Similarly, the two sets of subtree data are linearly equivalent.

Overview of the Proof of Crew's Conjecture

1. Compute the matrices of coefficients

$$X = [c_\lambda(T)]_{T \in \mathcal{T}_n,\ \lambda \vdash n} \qquad G = [g_T(a,b,c)]_{T \in \mathcal{T}_n,\ \lambda \vdash n}$$

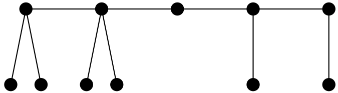
   for $n \leq 7$ or so.

2. Solve the matrix equation $X\Omega = G$ for Ω (there will be a large solution space).
3. Find a ~~needle~~ matrix Ω in the ~~haystack~~ solution space whose entries have a predictable combinatorial form.
4. Finish the proof (which mixes algebra and combinatorics).

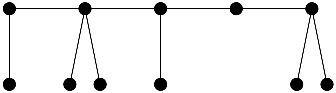
The proof of equivalence of the two subtree data sets is analogous.

Caterpillars

A tree is a **caterpillar** if deleting all its leaves produces a path.



C(3, 3, 1, 2, 2)



C(2, 3, 2, 1, 3)

Caterpillars are indexed by compositions with both first and last parts > 1, up to reversal.

Eisenstat and Gordon conjectured that for gap-free polynomials $p(x)$, the caterpillars arising from $(a+bx)p(x)$ and $(b+ax)p(x)$ have the same edge/leaf subtree data.

$$(2+1x)(1+x+x^3) = \mathbf{2} + 3x + x^2 + 2x^3 + \mathbf{1}x^4 \rightsquigarrow (\mathbf{3}, 3, 1, 2, \mathbf{2})$$
$$(1+2x)(1+x+x^3) = \mathbf{1} + 3x + 2x^2 + x^3 + \mathbf{2}x^4 \rightsquigarrow (\mathbf{2}, 3, 2, 1, \mathbf{3})$$

Caterpillars and Unique Factorization

For compositions $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_m)$, define

$$\alpha \cdot \beta = (a_1, \ldots, a_k, b_1, \ldots, b_m)$$
$$\alpha \odot \beta = (a_1, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_m)$$
$$\alpha \circ \beta = \beta^{\odot a_1} \cdot \beta^{\odot a_2} \cdots \beta^{\odot a_k}$$

Example

$(2,1) \circ (2,1) = (2,1)^{\odot 2} \cdot (2,1)^{\odot 1} = (2,3,1) \cdot (2,1) = (2,3,1,2,1)$
$(2,1) \circ (1,2) = (1,2)^{\odot 2} \cdot (1,2)^{\odot 1} = (1,3,2) \cdot (1,2) = (1,3,2,1,2)$

Fact [Billera–Thomas–van Willigenburg 2006]

Every composition α admits a unique irreducible factorization

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k.$$

The Eisenstat-Gordon Conjecture

Theorem [Aliste-Prieto, JLM, Wagner, Zamora 2024]

Reversing any of the irreducible factors in $1 \odot \alpha \odot 1$ produces a caterpillar with the same edge/leaf subtree counts.

For example,

$$\begin{cases}(2,1)\circ(2,1) = (\mathbf{2},3,1,2,\mathbf{1}) \\ (2,1)\circ(1,2) = (\mathbf{1},3,2,1,\mathbf{2})\end{cases} \implies \mathsf{C}(\mathbf{3},3,1,2,\mathbf{2}) = \mathsf{C}(\mathbf{2},3,2,1,\mathbf{3}).$$

In particular, if α has k irreducible factors then $\mathsf{C}(1 \odot \alpha \odot 1)$ is one of at least 2^{k-1} non-isomorphic caterpillars with the same subtree polynomial.

The case $k = 2$ implies the Eisenstat-Gordon conjecture.

Further Questions

Question 1: Does factorization extend from caterpillars to more general trees?

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Question 2: It is natural to consider the combined subtree data

$$\{\#\text{subtrees } S \text{ with } q \text{ edges, } r \text{ external edges, } s \text{ leaves}\colon q, r, s \in \mathbb{N}\}.$$

- This three-parameter subtree data is **strictly stronger** than either two-parameter data set
- Does it distinguish trees?
- Can we recover it from the CSF? (If so, not linearly.)

Thank you!

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Okay, You Asked For It

$$\omega(\lambda, a, b, c) = (-1)^{n-b-1} \sum_{\mu \vdash a} \binom{a - \ell(\mu)}{c} \binom{\lambda}{\mu} \binom{n - \ell(\lambda) + \ell(\mu) - a}{n - b - c - 1}$$

where $\ell(\lambda)$ means the length (= number of parts) of λ, and

$$\binom{\lambda}{\mu} := \prod_{i=1}^{n} \binom{\#\text{ of parts of } \lambda \text{ equal to } i}{\#\text{ of parts of } \mu \text{ equal to } i}$$