Simplicial Effective Resistance and Enumeration of Spanning Trees

Art Duval (University of Texas, El Paso)
Woong Kook (Seoul National University)
Kang-Ju Lee (Seoul National University)
Jeremy L. Martin (University of Kansas)

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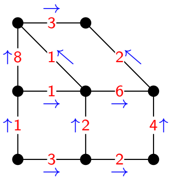
Resistor networks

A **[resistor] network** $N = (V, E, \mathbf{r})$ is a connected, undirected* graph (V, E) together with positive **resistances** $\mathbf{r} = (r_e)_{e \in E}$.



State of N:

currents $\mathbf{i} = (i_e)_{e \in E}$

voltages $\mathbf{v} = (v_e)_{e \in E}$

Ohm's law $$i_e r_e = v_e \quad (\forall e \in E)$$

Kirchhoff's current law $$\sum_{e \in E^{\text{in}}(x)} i_e - \sum_{e \in E^{\text{out}}(x)} i_e = 0 \quad (\forall x \in V)$$

Kirchhoff's voltage law $$\sum_{\vec{e} \in C} v_e = 0 \quad (\forall \text{ cycle } C)$$

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*Edges oriented for reference purposes only.

Effective resistance

Idea: Attach a **current generator**: an edge $\mathbf{e} = \overrightarrow{xy}$ with current $i_{\mathbf{e}}$, then look for currents and voltages satisfying OL, KCL, KPL.

Dirichlet principle The state of the system is the unique minimizer of "total energy" $\sum_e v_e i_e$ subject to OL, KCL, KPL.

Rayleigh principle As far as the external world is concerned, the system is equivalent to a single edge $\mathbf{e}$ with resistance

$$R_{\mathbf{e}}^{\text{eff}} = R_{xy}^{\text{eff}} = \frac{p_y - p_x}{c_{\mathbf{e}}}.$$

(the **effective resistance** of $\mathbf{e}$).

Effective resistance and tree counting

Theorem [Thomassen 1990]
Let $N = (V, E, \mathbf{r})$ be a network and $e = xy \in E$.

- If $\mathbf{r} \equiv 1$, then

$$R^{\text{eff}}_{xy} = \frac{\tau(G/xy)}{\tau(G)} = \frac{|\mathcal{T}(G/xy)|}{|\mathcal{T}(G)|}$$

where $\mathcal{T}(G)$ is the set of **spanning trees** of G.

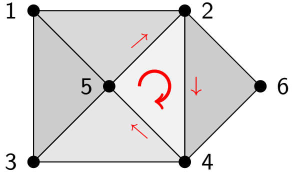
- More generally,

$$R^{\text{eff}}_{xy} = \frac{\hat{\tau}(G/xy)}{\hat{\tau}(G)} = \frac{\displaystyle\sum_{T \in \mathcal{T}(G/xy)} \prod_{e \in T} r_e^{-1}}{\displaystyle\sum_{T \in \mathcal{T}(G)} \prod_{e \in T} r_e^{-1}}.$$

Combinatorial application: weighted tree enumeration!

Simplicial complexes

- **Geometric simplicial complex**: family of simplices (points, line segments, triangles, tetrahedra, ...) attached along faces
- **Combinatorial simplicial complex:** $\Delta \subseteq 2^V$ such that $\sigma \in \Delta,\ \tau \subseteq \sigma \implies \tau \in \Delta$



$\langle 125, 135, 245, 345, 246 \rangle$

- Facets = maximal faces (denoted by Φ)
- Assume Δ^d **pure**: $|\phi| = d + 1$ for all facets ϕ

Boundary map and homology groups

Boundary of a k-simplex $\sigma = (v_0 < v_1 < \cdots < v_k)$:

$$\partial_k(v_0 < v_1 < \cdots < v_k) = \sum_{i=0}^{k} (-1)^i (v_0 \cdots \widehat{v_i} \cdots v_k)$$

Extending linearly gives a map

$$\partial_k : C_k(\Delta; R) \to C_{k-1}(\Delta; R)$$

where $C_k(\Delta; R)$ = linear comb'ns of k-simplices ($R = \mathbb{R}$ or $\mathbb{Z}$)

- **Key fact:** $\partial_k \circ \partial_{k+1} = 0$.

Homology: $H_k(\Delta; R) = \ker \partial_k / \operatorname{im} \partial_{k+1}$ (topological invariants)

- Homology groups are topological invariants of Δ

Spanning trees of simplicial complexes

A **spanning tree** of Δ^d is a subcomplex $\Upsilon \subset \Delta$ such that:

1. Υ contains all non-maximal faces (**spanning**)
2. $H_d(\Upsilon; \mathbb{R}) = 0$ (**acyclic**)
3. $H_{d-1}(\Upsilon; \mathbb{R}) = 0$ (**connected**)
   - Equivalent condition: $H_{d-1}(\Upsilon; \mathbb{Z})$ is *finite*

Examples:

- $d = 1$: standard definition of spanning tree of a graph
- Δ = simplicial sphere: remove a facet
- Δ = bubble wrap: pop all the bubbles (don't tear the sheet!)

Counting simplicial spanning trees

The right way to count simplicial trees:

$$\tau(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon; \mathbb{Z})|^2 \qquad \text{(unweighted)}$$

$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon; \mathbb{Z})|^2 \prod_{\phi \in \Upsilon} x_\phi \qquad \text{(unweighted)}$$

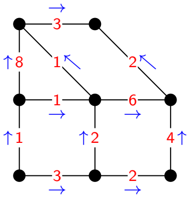
Kalai 1983: $\tau(K_{n_d}) = n^{\binom{n-2}{d}}$ using simplicial Laplacian $\partial\partial^{\mathrm{tr}}$.
(torsion factors arise naturally from Binet-Cauchy expansion)

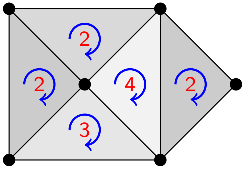
Subsequent work: Adin 1992 (complete colorful complexes), Petersson, Duval–Klivans–JLM, Lyons, Catanzaro–Chernyak–Klein (all c. 2006–2010)

Simplicial networks

Simplicial network: pure d-complex with resistances $(r_\phi)_{\phi\in\Phi}$

$d = 1$





$d = 2$

Currents $\mathbf{i} = (i_\phi)_{\phi\in\Phi}$ Voltages $\mathbf{v} = (v_\phi)_{\phi\in\Phi}$

Ohm's law	$i_\phi r_\phi = v_\phi$ for all $\phi \in \Phi$
Kirchhoff's current law	$\mathbf{i} \in \ker(\partial_d)$
Kirchhoff's voltage law	$\mathbf{v} \in \ker(\partial_d)^\perp$

Dirichlet, Rayleigh, R^{eff} have natural simplicial analogues.

Counting simplicial trees via effective resistance

Theorem [Kook–Lee 2018]
Let $(\Delta, \mathbf{r})$ be a simplicial network and σ a current generator. Then:

$$R_\sigma^{\text{eff}} = \frac{\hat{\tau}(\Delta/\sigma)}{\hat{\tau}(\Delta)} = \frac{\displaystyle\sum_{T \in \mathcal{T}(\Delta/\sigma)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\phi \in T} r_\phi^{-1}}{\displaystyle\sum_{T \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\phi \in T} r_\phi^{-1}}.$$

- Generalizes Thomassen's theorem for R^{eff} in graphs
- $\Delta/\sigma =$ quotient space (not simplicial, but close enough)
- Application: count trees by induction on facets

Shifted complexes

A simplicial complex on vertices $\{1, \ldots, n\}$ is **shifted** if any vertex of a face may be replaced with a smaller vertex.

Ex: threshold graphs; $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$

Duval–Klivans–JLM '09: recursion for $\hat{\tau}(\Delta)$ via the shifted complexes $\langle \phi \in \Delta \mid 1 \in \phi \rangle$ and $\langle \phi \in \Delta \mid 1 \notin \phi \rangle$; induct on n

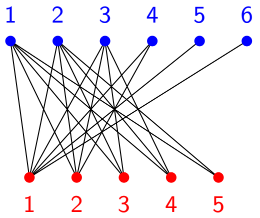
Duval–Kook–Lee–JLM '21$^+$: calculated R^{eff} for a shifted-maximal face; to obtain formula for $\hat{\tau}(\Delta)$, induct on $|\Phi|$

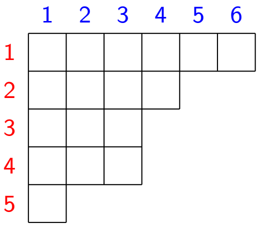
Color-shifted complexes

A simplicial complex Δ^d is **color-shifted** [Babson–Novik '06] if:

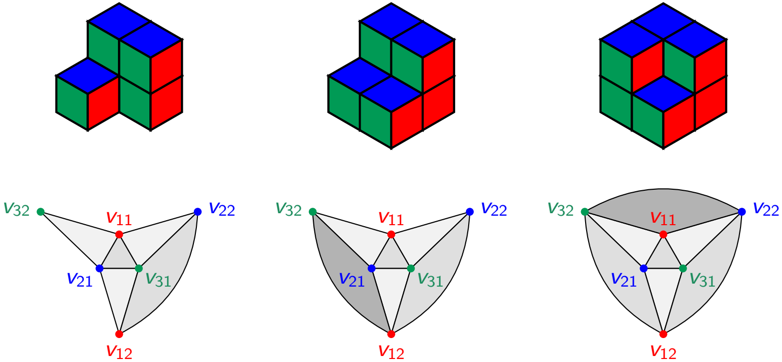
- $V(\Delta) = V_1 \cup \cdots \cup V_{d+1}$, where $V_q = \{v_{q1}, \ldots, v_{q\ell_q}\}$
- Each facet contains exactly one vertex of each color
- A vertex may be replaced with a smaller vertex of same color

$d = 1$: *Ferrers graphs* [Ehrenborg–van Willigenburg '04]





Color-shifted complexes



Trees in color-shifted complexes

Vertex-weighted spanning tree enumerators:

$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{\phi \in \Upsilon} \prod_{v_{qj} \in \phi} x_{qj}$$
$$= \sum_{\Upsilon \in \mathcal{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{q,j} x_{qj}^{\deg_\Upsilon(v_{qj})}$$

Proposition [Duval–Kook–Lee–JLM 2021$^+$]
Let Δ^d color-shifted, $\sigma = v_{1,k_1} v_{2,k_2} \dots v_{d+1,k_{d+1}} \notin \Delta$.
Then:

$$R^{\text{eff}}(\sigma) = \frac{\hat{\tau}(\Delta \cup \sigma)}{\tau(\Delta)} = \prod_{q=1}^{d+1} \frac{x_{q,1} + \cdots + x_{q,k_q}}{x_{q,1} + \cdots + x_{q,k_q - 1}}.$$

Trees in color-shifted complexes

Theorem [Duval–Kook–Lee–JLM 2021^{+}]

$$\hat{\tau}(\Delta) = \prod_{q,i} x_{q,i}^{e(q,i)} \prod_{\substack{\rho \in \Delta \\ \dim \rho = d-1}} (x_{m(\rho),1} + \cdots + x_{m(\rho),k(\rho)})$$

where

$$\begin{aligned} e(q,i) &= \#\{\sigma \in \Delta_d \mid v_{q,i} \in \sigma \text{ and } v_{q',1} \in \sigma \text{ for some } q' \neq q\} \\ m(\rho) &= \text{unique color missing from } \rho \\ k(\rho) &= \max\{j \mid \rho \cup v_{m(\rho),j} \in \Delta\} \end{aligned}$$

- Previously conjectured by Aalipour and Duval [unpublished]
- Result seems inaccessible without effective resistance

Thank you!