Spanning Trees of Simplicial Complexes

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RMMC 2011
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Le Menu

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1. **Appetizer:** Graphs
   - The incidence and Laplacian matrices
   - The matrix-tree theorem
   - The chip-firing game
   - The critical group

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   - The incidence and Laplacian matrices
   - The matrix-tree theorem
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2. **Main Course:** Simplicial Complexes
   - Crash course in algebraic topology
   - Simplicial spanning trees
   - Simplicial matrix-tree theorems
   - Simplicial critical groups

- Main course is joint work with Art Duval (U. of Texas, El Paso) and Caroline Klivans (U. of Chicago)

Appetizer: Graphs

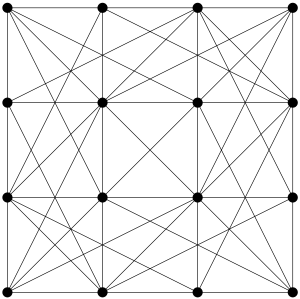
Spanning Trees

Definition A **spanning tree** of a graph $G = (V, E)$ is a set of edges T (or, equivalently, a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

- Any two of these conditions together imply the third.

Spanning Trees



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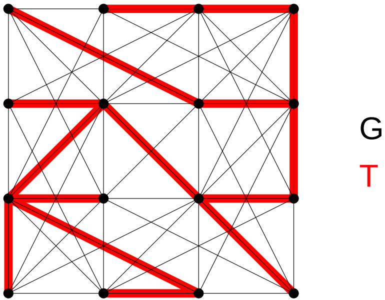

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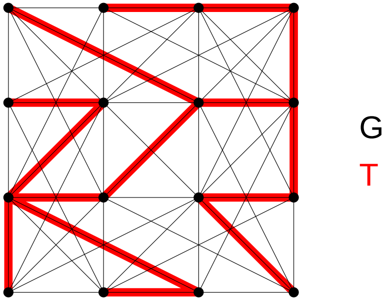

Counting Spanning Trees

$\tau(G)$ = number of spanning trees of G

- $\tau(\text{tree}) = 1$
- $\tau(n\text{-cycle}) = n$

- Complete graph: $\tau(K_n) = n^{n-2}$ (Cayley's formula)
- Complete bipartite graph: $\tau(K_{n,m}) = n^{m-1}m^{n-1}$
- Many other enumeration formulas for nice graphs (threshold graphs, hypercubes, ...)

The Incidence Matrix

Definition **(Signed) incidence matrix** ∂ of G

- Rows indexed by vertices; columns indexed by edges
- Each column has one 1 and one -1 corresponding to its endpoints, and 0s elsewhere.

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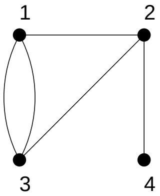
$G =$ 

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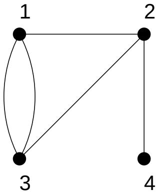
$G =$ 

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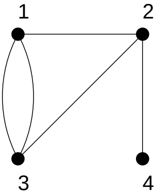
$G =$ 

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- If we can count column bases, we can count spanning trees.

The Laplacian Matrix

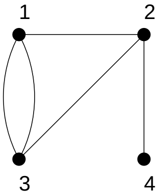
Definition The **Laplacian matrix** of G is $L = \partial\partial^T$.

Entries of L are scalar products of rows of ∂:

$$L_{(i,j)} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -(\#\text{ of edges joining } i \text{ and } j) & \text{otherwise.} \end{cases}$$

- $\operatorname{rank} L = \operatorname{rank} \partial = \#$ vertices $- \#$ components.

The Laplacian Matrix



$$\partial = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad L = \begin{pmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

The Matrix-Tree Theorem

The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} \ .$$

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(2) Let $1 \leq i \leq n$. Form the *reduced Laplacian* $\tilde{L}$ by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det \tilde{L} \ .$$

The Matrix-Tree Theorem

Sketch of proof: By the Binet-Cauchy formula from linear algebra,

$$\mathbf{det}\ \tilde{\mathbf{L}} = \det \tilde{\partial}\tilde{\partial}^T = \sum_{\substack{A \subseteq E \\ |A|=n-1}} (\det \tilde{\partial}_A)^2 \qquad (\tilde{\partial}\text{: delete a row from } \partial)$$

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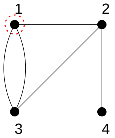
$= \text{number of column bases of } \partial$

$=$ **number of spanning trees!**

The Matrix-Tree Theorem

Example

$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \tilde{L} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues: 0, 1, 4, 5

$(1 \cdot 4 \cdot 5)/4 = \mathbf{5}$

$\det \tilde{L} = \mathbf{5}$

The Matrix-Tree Theorem

Example $G = K_n$ (complete graph on n vertices)

$$L(K_n) = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix}$$

- Eigenvalues: 0 (multiplicity 1), n (multiplicity $n-1$)
- $\tau(K_n) = n^{n-1}/n = \mathbf{n}^{n-2}$.

Example: The Hypercube

- $G = Q_n =$ 1-skeleton of n-dimensional hypercube
- Eigenvalues of L: $0, 2, 4, \ldots, 2n$, with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

$$\Longrightarrow \quad \tau(Q_n) = \prod_{k=2}^{n} (2k)^{\binom{n}{k}}.$$

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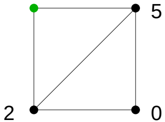
Open Problem Find a bijective proof of this formula.

The Chip-Firing Game



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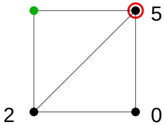

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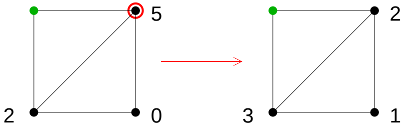

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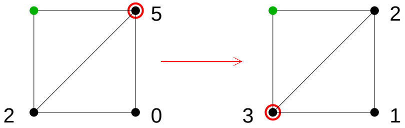

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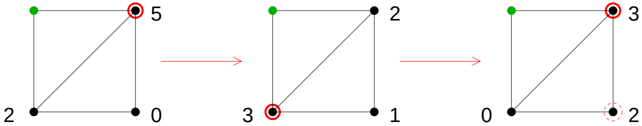

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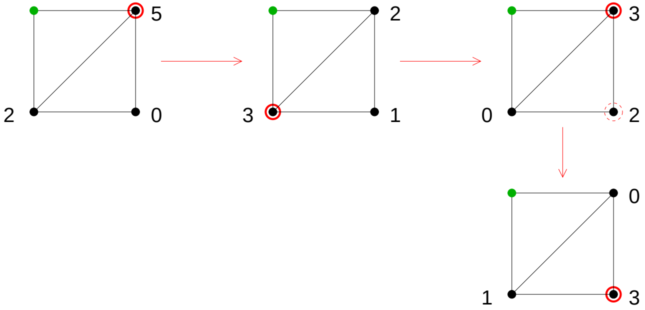

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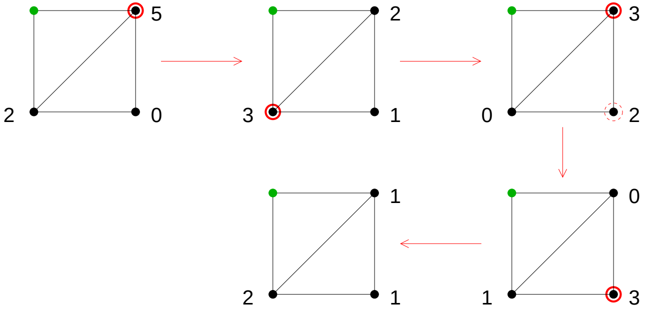

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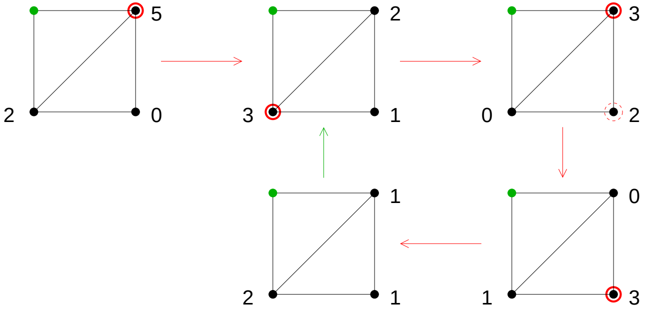

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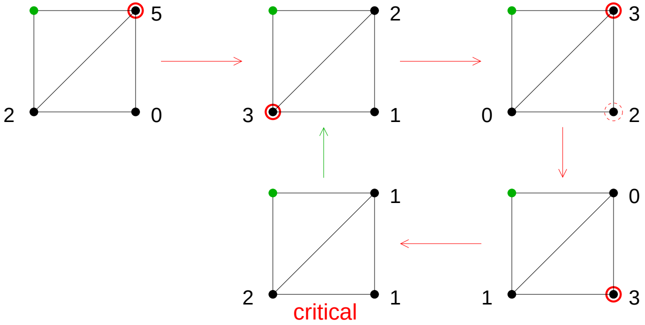

The Chip-Firing Game

- G: graph with vertex set $\{1, 2, \ldots, n\}$
- Each vertex $i < n$ has a finite number c_i of poker chips
- A vertex **fires** by giving one chip to each of its neighbors
- Vertex n, the **bank**, only fires if no other vertex can fire
- Vertices other than the bank cannot go into debt
- **Chip configuration** = vector $\mathbf{c} = (c_1, \ldots, c_{n-1}) \in \mathbb{N}^{n-1}$

The Chip-Firing Game

Theorem (Biggs, Dhar?, Björner–Lovász–Shor)
Every initial chip configuration determines a unique critical configuration, regardless of the order of firing.

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Recall that the Laplacian matrix of G is $L = [\ell_{ij}]_{1 \leq i,j \leq n}$ where

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -(\# \text{ of edges joining } i \text{ and } j) & \text{otherwise.} \end{cases}$$

- Firing vertex $i \longleftrightarrow$ subtracting i^{th} column of L from $\mathbf{c}$.

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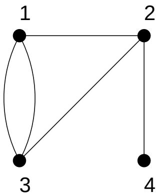
Definition The **critical group** of G is

$$K(G) = \mathbb{Z}^{n-1} / \operatorname{colspace}(\tilde{L}).$$

- $|K(G)| = \tau(G)$ by Matrix-Tree Theorem
- Critical configurations are a system of coset representatives

Cuts and Flows

$G =$



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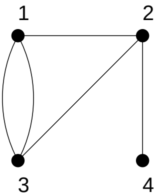
$G =$ 

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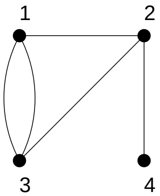
$G =$ 

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Flow space $\mathcal{F} = \ker(\partial) = \mathcal{C}^\perp$ (generated by cycles)

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Theorem [Bacher, de la Harpe, Nagnibeda 1997]

$$K(G) \ = \ \mathbb{Z}^{n-1} / \,\text{colspace}\,\tilde{L} \ \cong \ \mathbb{Z}^E / (\mathcal{C} \oplus \mathcal{F}).$$

Main Course: Simplicial Complexes

Simplicial Complexes

Definition A **simplicial complex** is a family $\Delta \subseteq \text{powerset}(\{1, 2, \ldots, n\})$ such that

$$\text{if } \sigma \in \Delta \text{ and } \sigma' \subseteq \sigma, \text{ then } \sigma' \in \Delta.$$

- **Think of a simplicial complex as a higher-dimensional generalization of a graph.**

- Elements of Δ are called *faces* or *simplices*.
- $\dim \sigma = |\sigma| - 1$
- $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$
- $f_i(\Delta) =$ number of i-dimensional faces of Δ

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- Simplicial polytopes (minus geometry)
- Every "reasonable" topological space can be represented as a simplicial complex
- Graphs = 1-dimensional simplicial complexes
- Simplicial complexes arise frequently in combinatorics: e.g., order complexes of posets

A Crash Course in Algebraic Topology

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Definition The **$\mathbf{i^{th}}$ (reduced) homology group** of Δ is

$$\begin{aligned}\tilde{H}_i(\Delta) &= \ker(\partial_i) \;/\; \operatorname{im}(\partial_{i+1}) \\ &\cong \mathbb{Z}^{\tilde{\beta}_i(\Delta)} \oplus \text{finite "torsion" group}\end{aligned}$$

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(If you're new at this: Don't worry about the twiddles!)

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- If Δ is a d-sphere, then

$$\tilde{H}_i(\Delta) = \begin{cases} \mathbb{Z} & \text{for } i = d, \\ 0 & \text{for } i < d. \end{cases}$$

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- *Case 2:* Tear a $(d-1)$-dimensional hole: $\tilde{\beta}_{d-1}$ increases by 1

- **Fact** The (reduced) Euler characteristic of Δ is

$$\tilde{\chi}(\Delta) = \sum_i (-1)^i f_i(\Delta) = \sum_i (-1)^i \tilde{\beta}_i(\Delta).$$

Simplicial Spanning Trees

Definition Let Δ be a simplicial complex of dimension d.

A **simplicial spanning tree (SST)** is a subcomplex $\Upsilon \subset \Delta$, with $\Upsilon_{(d-1)} = \Delta_{(d-1)}$, such that

1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group;
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- Any two of conditions 1,2,3 together imply the third (just as for graphs).

Examples of SSTs

What if Δ is a simplicial d-sphere?

- Recall that $\tilde{H}_d(\Delta) = \mathbb{Z}$. To make $\tilde{H}_d(\Upsilon) = 0$, "pop the bubble" by deleting a single facet from Δ. (But don't delete more than one or $\tilde{H}_{d-1}$ will become nonzero.)
- In particular, # of SSTs = # facets = $f_d(\Delta)$. (Analogous to the statement that the spanning trees of a cycle graph are formed by deleting a single edge.)

Kalai's Theorem

Let K_n^d be the d-skeleton of the n-vertex simplex, i.e.,

$$K_n^d = \left\{ F \subseteq \{1, 2, \ldots, n\} \;\mid\; \dim F \leq d \right\}$$

and let $\mathcal{T}(\Delta)$ denote the set of SSTs of Δ.

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Theorem [Kalai 1983]

$$\sum_{\Upsilon \in \mathcal{T}(K_n^d)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 \;=\; n^{\binom{n-2}{d}}.$$

- Setting $d = 1$ recovers Cayley's formula $\tau(K_n) = n^{n-2}$.

Counting Simplicial Spanning Trees

$\Delta = d$-dim'l simplicial complex with $|\tilde{H}_i(\Delta)| < \infty \ \forall i < d$

$L = \partial_d \partial_d^T$ (simplicial Laplacian)

$$\tau_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta_{(k)})} |\tilde{H}_{k-1}(\Upsilon)|^2 \quad (\text{"number" of } k\text{-dim'l trees"})$$

Simplicial Matrix-Tree Theorem I [Duval–Klivans–JLM 2007]

$$\tau_d(\Delta) \;=\; |\tilde{H}_{d-2}(\Delta)|^2 \cdot \frac{\text{product of nonzero eigenvalues of } L}{\tau_{d-1}(\Delta)}.$$

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$$\tau_k(\Delta) = \sum_{\Upsilon \in \mathcal{T}(\Delta_{(k)})} |\tilde{H}_{k-1}(\Upsilon)|^2$$

Γ = simplicial spanning tree of $\Delta_{(d-1)}$

L_Γ = reduced Laplacian obtained from $L = \partial_d \partial_d^T$ by deleting Γ

Simplicial Matrix-Tree Theorem II

$$\tau_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Gamma)|^2} \det L.$$

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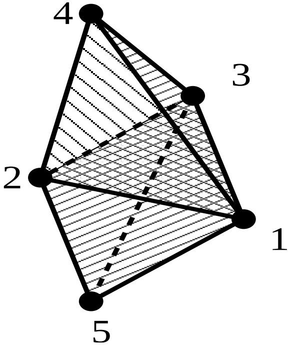
. . . but some trees may be more equal than others.

An Example: The Equatorial Bipyramid B



Facets: 123 ("equator")
124, 134, 234 ("northern")
125, 135, 235 ("southern")

$f(\Delta) = (5, 9, 7)$

$\tilde{H}_0(\Delta) = 0$
$\tilde{H}_1(\Delta) = 0$
$\tilde{H}_2(\Delta) = \mathbb{Z}^2$

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SMTT-II: Take $\Gamma = \{12, 13, 14, 15\}$; then $\det L_\Gamma = 15$.

Some Open Problems

Pick your favorite simplicial (or even cell) complex and count its spanning trees!

It helps if the complex is *Laplacian integral* (i.e., the Laplacian matrix has integer eigenvalues).

- Complete colorful complexes: Adin '92
- Shifted complexes: Duval–Reiner '03, weighted DKM '07
- Skeletons of cubes: DKM '10
- Matroid complexes: Kook–Reiner–Stanton '01; **weighted?**
- **Matching and chessboard complexes?**

Critical Groups of Simplicial Complexes

Critical group of a graph G:

$$K(G) = \text{coker}\,\tilde{L} = \text{coker}(\tilde{\partial}\tilde{\partial}^T) = \mathbb{Z}^{|E|}/(\mathcal{C} \oplus \mathcal{F})$$

where $\partial =$ incidence matrix; $\mathcal{C} = \text{colspace}\,\partial^T$; $\mathcal{F} = \ker \partial$.

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Theorem [DKM'10] $|K_{i-1}(\Delta)| = \tau_i(\Delta)$ for all i.

Simplicial Chip-Firing?

Open Problem

Develop a simplicial analogue of the chip-firing game whose critical configurations correspond to elements of the simplicial critical group.