Simplicial Matrix-Tree Theorems

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Graphs and Spanning Trees

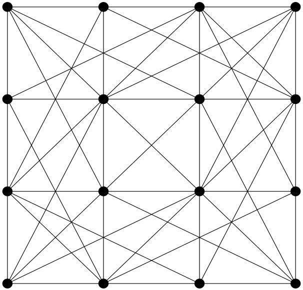
$G = (V, E)$: simple connected graph

Definition A **spanning tree of G** is a subgraph (V, T) such that

1. (V, T) is **connected** (every pair of vertices is joined by a path);
2. (V, T) is **acyclic** (contains no cycles);
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Graphs and Spanning Trees



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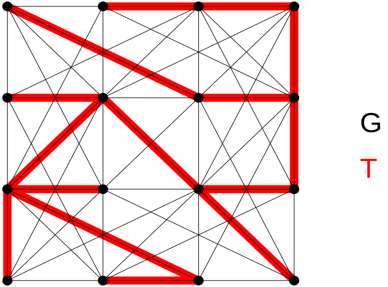
Graphs and Spanning Trees



Counting Spanning Trees

Let $\tau(G)$ denote the number of spanning trees of G.

Graph G	$\tau(G)$
Any tree	1
C_n (cycle on n vertices)	n
K_n (complete graph on n vertices)	n^{n-2} (Cayley)
$K_{p,q}$ (complete bipartite graph)	$p^{q-1}q^{p-1}$ (Fiedler-Sedláçek)
Q_n (n-dimensional hypercube)	$2^{2^n-k-1}\prod_{k=2}^{n} k^{\binom{n}{k}}$

The Laplacian Matrix

Let $G = (V, E)$ be a graph with $V = [n] = \{1, 2, \ldots, n\}$.

Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j, \\ -1 & \text{if } i, j \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

- L is a real symmetric matrix
- $L = MM^{tr}$, where M is the signed incidence matrix of G

The Matrix-Tree Theorem

Matrix-Tree Theorem, Version I: Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$

Matrix-Tree Theorem, Version II: Let L_i be the *reduced Laplacian* obtained by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i.$$

[Kirchhoff, 1847]

The Matrix-Tree Theorem

Sketch of proof:

1. Expand $\det L_i$ using the Binet-Cauchy formula:

$$\det L_i = \det M_i \, M_i^{tr} = \sum_{\substack{T \subset E \\ |T| = n-1}} (\det M_T)^2$$

where M_T = square submatrix of M_i with columns T

2. Show that

$$\det M_T = \begin{cases} \pm 1 & \text{if } T \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

Weighted Spanning Tree Enumerators

Idea: Let's record combinatorial information about a spanning tree T by assigning it a monomial weight x_T.

(e.g., vertex degrees; number of edges in specified sets; etc.)

Definition The **weighted spanning tree enumerator of G** is the generating function

$$\sum_{T \in \mathscr{T}(G)} x_T$$

where $\mathscr{T}(G)$ denotes the set of spanning trees of G.

Weighted Spanning Tree Enumerators

The weighted spanning tree enumerator of a graph

- reveals much more detailed combinatorial information about spanning trees of G than merely counting them
  - (particularly when it factors!)
- can suggest bijective proofs of formulas for $\tau(G)$

The Weighted Laplacian

Introduce an indeterminate e_{ij} for each pair of vertices i, j.
Set $e_{ij} = e_{ji}$, and if i, j are not adjacent, then set $e_{ij} = 0$.

The **weighted Laplacian of G** is the $n \times n$ matrix $\hat{L} = [\hat{\ell}_{ij}]$, where

$$\hat{\ell}_{ij} = \begin{cases} \sum_{j \neq i} e_{ij} & \text{if } i = j, \\ -e_{ij} & \text{if } i \neq j. \end{cases}$$

- Setting $e_{ij} = 1$ for each edge ij recovers the usual Laplacian L.

The Weighted Matrix-Tree Theorem

Weighted Matrix-Tree Theorem I: If $0, \hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{n-1}$ are the eigenvalues of $\hat{L}$, then

$$\sum_{T \in \mathscr{T}(G)} \prod_{ij \in T} e_{ij} = \frac{\hat{\lambda}_1 \hat{\lambda}_2 \cdots \hat{\lambda}_{n-1}}{n}.$$

Weighted Matrix-Tree Theorem II: If $\hat{L}_{k\ell}$ is obtained by deleting the k^{th} row and ℓ^{th} column of $\hat{L}$, then

$$\sum_{T \in \mathscr{T}(G)} \prod_{ij \in T} e_{ij} = (-1)^{k+\ell} \det \hat{L}_{k\ell}.$$

Example: The Cayley-Prüfer Theorem

Weight spanning trees of complete graph K_n by degree sequence:

$$x_T = \prod_{i=1}^{n} x_i^{\deg_T(i)}$$

Theorem [Cayley-Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_T = (x_1 x_2 \cdots x_n)(x_1 + x_2 + \cdots + x_n)^{n-2}.$$

(Setting $x_i = 1$ for all i recovers Cayley's formula.)

Example: The Cayley-Prüfer Theorem

Theorem [Cayley-Prüfer]

$$\sum_{T \in \mathscr{T}(K_n)} x_T = (x_1 x_2 \cdots x_n)(x_1 + x_2 + \cdots + x_n)^{n-2}.$$

Combinatorial proof: the *Prüfer code*, a bijection

$$P : \mathscr{T}(K_n) \to [n]^{n-2}$$

where $\deg_T(i) = 1 +$ number of i's in $P(T)$.

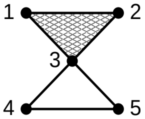
More Weighted Spanning Tree Enumerators

- $K_{p,q}$: degree sequence (bijection: Hartsfield–Werth)
- Threshold graphs: degree sequence and more (Remmel–Williamson)
- Ferrers graphs: degree sequence (Ehrenborg–van Willigenburg)
- Hypercubes: direction and facet degrees (JLM–Reiner; bijection??)

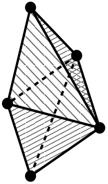
Simplicial Complexes

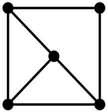
Definition A **simplicial complex on vertex set V** is a family Δ of subsets of V such that

- $\{v\} \in \Delta$ for every $v \in V$;
- If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.



{{}, 1, 2, 3, 4, 5,
12, 13, 23, 34, 35, 45,
123}

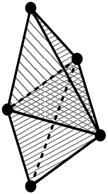




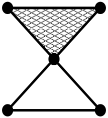
Simplicial Complexes

- Elements of Δ are called **faces**.
- Maximal faces are called **facets**.
- $\dim F = |F| - 1; \quad \dim \Delta = \max\{\dim F \mid F \in \Delta\}$.
- Δ is *pure* if all facets have equal dimension.
- $f_i(\Delta)$ = number of i-dimensional faces.
- The **k-skeleton** is $\Delta_{(k)} = \{F \in \Delta \mid \dim F \leq k\}$.
- A graph is just a simplicial complex of dimension 1.

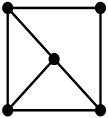
Simplicial Complexes



pure; dim=2
$f(\Delta)=(5,9,7)$



not pure; dim=2
$f(\Delta)=(5,6,1)$



pure; dim=1
$f(\Delta)=(5,7)$

Simplicial Homology

R = commutative ring with identity (typically $\mathbb{Z}$ or $\mathbb{Q}$)
$C_i(\Delta)$ = free R-module on i-dimensional faces of Δ

Δ has natural **boundary** and **coboundary maps**

$$\partial_i : C_i \to C_{i-1}, \qquad \partial_i^* : C_{i-1} \to C_i$$

such that

$$\partial_i \circ \partial_{i+1} = \partial_{i+1}^* \circ \partial_i^* = 0.$$

Simplicial Homology

Definition The i^{th} **reduced simplicial homology group** of Δ is

$$\tilde{H}_i(\Delta; R) = \ker \partial_i \,/\, \operatorname{im} \partial_{i+1}.$$

- Homology groups over $\mathbb{Q}$ measure the holes in Δ.
- Homology groups over $\mathbb{Z}$ measure holes (the free part) and "twisting" (the torsion part).

Definition The i^{th} **Betti number of Δ** is

$$\tilde{\beta}_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}_i(\Delta, \mathbb{Q}).$$

Simplicial Homology

Let G be a graph (a 1-dimensional simplicial complex).

- $\tilde{\beta}_0(G) =$ (number of connected components of G) $- 1$
- $\tilde{\beta}_1(G) =$ number of edges that need to be deleted to make G acyclic

- ∂_1 is the signed vertex-edge incidence matrix M.
- The Laplacian of G is $L = \partial_1 \partial_1^*$.

Simplicial Spanning Trees

Let Δ^d be a simplicial complex (i.e., $\dim \Delta = d$).

Let $\Upsilon \subset \Delta$ be a subcomplex with $\Upsilon_{(d-1)} = \Delta_{(d-1)}$.

Definition Υ is a **simplicial spanning tree** (SST) of Δ if

1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$;
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$.

Simplicial Spanning Trees

Conditions for $\Upsilon \subset \Delta^d$ to be an SST:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$ ("connected");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- Any two of these conditions together imply the third
- When $d = 1$, coincides with the usual definition

Metaconnectedness

Denote by $\mathscr{T}(\Delta)$ the set of simplicial spanning trees of Δ.

Proposition $\mathscr{T}(\Delta) \neq \emptyset$ if and only if Δ has the homology type of a wedge of d-spheres:

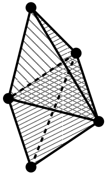
$$\tilde{\beta}_j(\Delta) = 0 \qquad \forall j < \dim \Delta.$$

Equivalently,

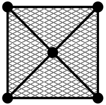
$$|\tilde{H}_j(\Delta; \mathbb{Z})| < \infty \qquad \forall j < \dim \Delta.$$

Such a complex is called **metaconnected**.

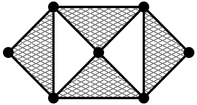
Metaconnectedness



metaconnected



metaconnected



not metaconnected

Metaconnectedness

- Every acyclic complex is metaconnected.
- Every Cohen-Macaulay complex is metaconnected (by Reisner's theorem), including:
  - 0-dimensional complexes
  - connected graphs
  - simplicial spheres
  - shifted complexes
  - matroid complexes
  - many other complexes arising in algebra and combinatorics

Examples of SSTs

Example If $\dim \Delta = 0$, then $\mathscr{T}(\Delta) = \{\text{vertices of } \Delta\}$.

Example If Δ is $\mathbb{Q}$-acyclic, then $\mathscr{T}(\Delta) = \{\Delta\}$.

- Includes complexes that are not $\mathbb{Z}$-acyclic, such as $\mathbb{RP}^2$.

Example If Δ is a simplicial sphere, then

$$\mathscr{T}(\Delta) = \{\Delta \setminus \{F\} \mid F \text{ a facet of } \Delta\}.$$

- Simplicial spheres are the analogues of cycle graphs.

Kalai's Theorem

Let Δ be the d-skeleton of the n-vertex simplex:

$$\Delta = \{F \subset [n] \mid \dim F \leq d\}.$$

Theorem [Kalai 1983]

$$\sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

- Reduces to Cayley's formula when $d = 1$ ($\Delta = K_n$).
- Adin (1992): Analogous formula for *complete colorful complexes* (generalizing Fiedler-Sédlaçek formula for $K_{p,q}$)

Simplicial Analogues of Graph Invariants

Let Δ^d be a metaconnected simplicial complex.

$$C_{i-1}(\Delta) \xleftarrow{\partial_i^*} C_i(\Delta) \xrightarrow{\partial_i} C_{i-1}(\Delta)$$

$L_i = \partial_i \partial_i^*$ (the **up-down Laplacian**)

$s_i =$ product of nonzero eigenvalues of L_i

$$h_i = \sum_{\Upsilon \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(\Upsilon)|^2$$

The Simplicial Matrix-Tree Theorem — Version I

s_i = product of nonzero eigenvalues of Laplacian L_i

$$h_i = \sum_{\Upsilon \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(\Upsilon)|^2$$

Theorem [Duval–Klivans–JLM 2006]

$$h_d \;=\; \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2.$$

Special Cases

- When Δ is a graph on n vertices, the theorem says that

$$h_1 = \frac{s_1}{h_0}|\tilde{H}_{-1}(\Delta)|^2 = \frac{s_1}{n}$$

which is the classical Matrix-Tree Theorem.

- If $\tilde{H}_i(\Delta, \mathbb{Z}) = 0$ for $i \leq d-2$, then

$$h_d = \frac{s_d \, s_{d-2} \, \cdots}{s_{d-1} \, s_{d-3} \, \cdots}$$

The Simplicial Matrix-Tree Theorem — Version II

Δ^d = simplicial complex

$\Gamma \in \mathscr{T}(\Delta_{(d-1)})$

∂_Γ = restriction of ∂_d to faces not in Γ

$L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [Duval–Klivans–JLM 2006]

$$h_d = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma.$$

Simplicial Matrix-Tree Theorems

Theorem (SMTT–I: product of eigenvalues)

$$h_d \;=\; \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2$$

Theorem (SMTT–II: reduced Laplacian)

$$h_d \;=\; \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma$$

- Version II is more useful for computing h_d directly.
- In many cases, the $\tilde{H}_{d-2}$ terms are trivial.

Weighted SST Enumeration

- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply the F^{th} column of ∂ by x_F
- Weighted Laplacian $\mathbf{L} = \boldsymbol{\partial}\boldsymbol{\partial}^*$
- Weighted analogues of s_i and h_i :

$$\mathbf{s}_i = \text{product of nonzero eigenvalues of } \mathbf{L}_i$$

$$\mathbf{h}_i = \sum_{T \in \mathscr{T}(\Delta_{(i)})} |\tilde{H}_{i-1}(T)|^2 \prod_{F \in T} x_F^2$$

Weighted Simplicial Matrix-Tree Theorems

Weighted Simplicial Matrix-Tree Theorem I

$$\mathbf{h}_i = \frac{\mathbf{s}_i}{\mathbf{h}_{i-1}} \; |\tilde{H}_{i-2}(\Delta)|^2$$

Weighted Simplicial Matrix-Tree Theorem II

$$\mathbf{h}_i \;=\; \frac{|\tilde{H}_{i-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{i-2}(\Gamma;\mathbb{Z})|^2} \; \det \mathbf{L}_\Gamma$$

where $\Gamma \in \mathscr{T}(\Delta_{(i-1)})$

Weighted SST Enumeration

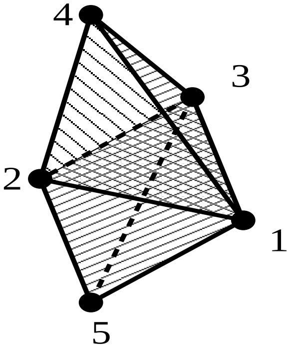
As in the graphic case, we can use weights to obtain finer enumerative information about simplicial spanning trees.

In order for the weighted simplicial spanning tree enumerators to factor, we need **L** to have integer eigenvalues.

That is, Δ must be **Laplacian integral**.

- Shifted complexes
- Matroid complexes
- Others?

Example: The Bipyramid With Equator



Vertices: 1, 2, 3, 4, 5

Edges: All but 45

Facets: 123, 124, 134, 234, 125, 135, 235

$f(\Delta) = (5, 9, 7)$

"Equator": the facet 123

Example: The Bipyramid With Equator

$$\begin{aligned}\Delta &= \text{bipyramid with equator} \\ &= \langle 123, 124, 134, 234, 125, 135, 235 \rangle\end{aligned}$$

- For each facet $F = ijk$, set $x_F = x_i x_j x_k$.

Enumeration of SSTs of Δ by degree sequence:

$$\begin{aligned}\mathbf{h}_2 &= \sum_{T \in \mathscr{T}(\Delta)} \prod_{i \in V} x_i^{\deg_T(i)} \\ &= x_1^3 x_2^3 x_3^3 x_4^2 x_5^2 (x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)\end{aligned}$$

Shifted Complexes

Definition A simplicial complex Δ on vertices $[n]$ is **shifted** if

for all $F \in \Delta$, $i \in \Delta$, $j \notin \Delta$, and $j < i$,

we have $F \setminus \{i\} \cup \{j\} \in \Delta$.

Example If Δ is shifted and $235 \in \Delta$, then Δ must also contain the faces 234, 135, 134, 125, 124, 123.

- Shifted complexes of dimension 1 are *threshold graphs*.

Shifted Complexes

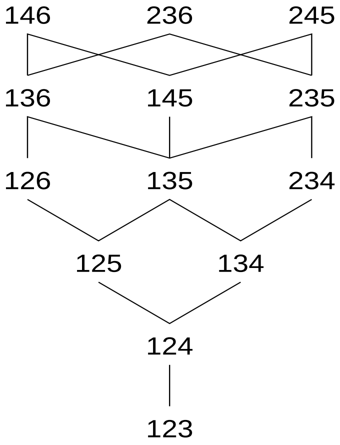
Define the **componentwise (partial) order** on $(d+1)$-sets of positive integers

$$A = \{a_1 < a_2 < \cdots < a_{d+1}\},$$
$$B = \{b_1 < b_2 < \cdots < b_{d+1}\}$$

by

$$A \preceq B \quad \iff \quad a_i \leq b_i \text{ for all } i.$$

- The set of facets of a shifted complex is a *lower order ideal* with respect to $\preceq$.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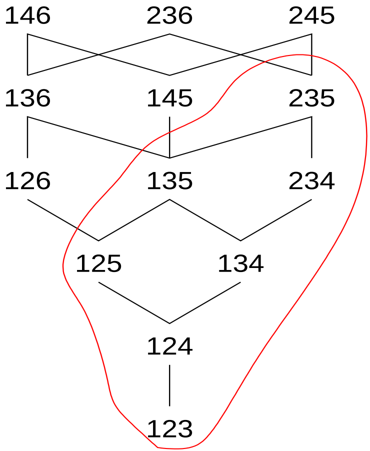
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Shifted Complexes

Proposition Shifted complexes are shellable, hence Cohen-Macaulay, hence metaconnected.

Theorem [Duval–Reiner 2001]
For Δ shifted, the eigenvalues of the unweighted Laplacian L are given by the transpose of the vertex/facet degree sequence.

- In particular, shifted complexes are Laplacian integral.

The Combinatorial Fine Weighting

Let Δ^d be a shifted complex on vertices $[n]$.
For each facet $A = \{a_1 < a_2 < \cdots < a_{d+1}\}$, define

$$x_A = \prod_{i=1}^{d+1} x_{i,a_i} \ .$$

Example If $\Upsilon = \langle 123, 124, 134, 135, 235 \rangle$ is a simplicial spanning tree of Δ, its contribution to h_2 is

$$x_{1,1}^4 x_{1,2} x_{2,2}^2 x_{2,3}^3 x_{3,3} x_{3,4}^2 x_{3,5}^2 \ .$$

The Algebraic Fine Weighting

For faces $A \subset B \in \Delta$ with $\dim A = i - 1$, $\dim B = i$, define

$$X_{AB} = \frac{\uparrow^{d-i} x_B}{\uparrow^{d-i+1} x_A}$$

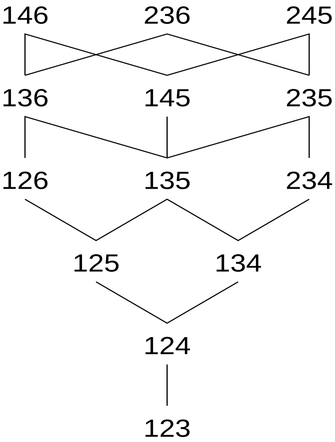
where $\uparrow x_{i,j} = x_{i+1,j}$.

- Weighted boundary maps ∂ satisfy $\partial\partial = 0$.
- Laplacian eigenvalues are the same as those for the combinatorial fine weighting, except for denominators.

Critical Pairs

Definition A **critical pair** of a shifted complex Δ^d is an ordered pair (A, B) of $(d+1)$-sets of integers, where

- $A \in \Delta$ and $B \notin \Delta$; and
- B covers A in componentwise order.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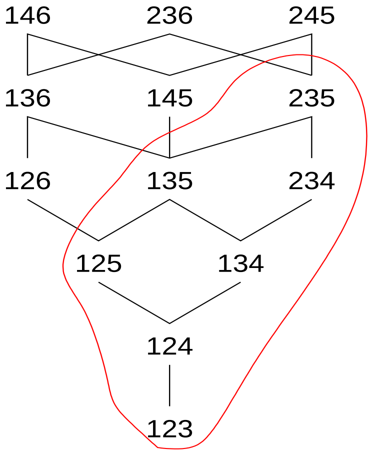
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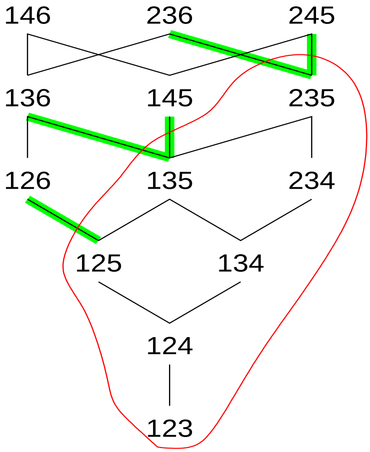
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The Signature of a Critical Pair

Let (A, B) be a critical pair of a complex Δ:

$$A = \{a_1 < a_2 < \cdots < a_i < \cdots < a_{d+1}\},$$
$$B = A \setminus \{a_i\} \cup \{a_i + 1\}.$$

Definition The **signature** of (A, B) is the ordered pair

$$\big(\{a_1, a_2, \ldots, a_{i-1}\},\ a_i\big).$$

Finely Weighted Laplacian Eigenvalues

Theorem [Duval–Klivans–JLM 2007]

Let Δ^d be a shifted complex.

Then the finely weighted Laplacian eigenvalues of Δ are specified completely by the signatures of critical pairs of Δ.

$$\text{signature } (S, a) \quad \Longrightarrow \quad \text{eigenvalue } \frac{1}{\uparrow X_S} \sum_{j=1}^{a} X_{S \cup j}$$

Examples of Finely Weighted Eigenvalues

- Critical pair (135,145); signature (1,3):

$$\frac{X_{11}X_{21} + X_{11}X_{22} + X_{11}X_{23}}{X_{21}}$$

- Critical pair (235,236); signature (23,5):

$$\frac{X_{11}X_{22}X_{33} + X_{12}X_{22}X_{33} + X_{12}X_{23}X_{33} + X_{12}X_{23}X_{34} + X_{12}X_{23}X_{35}}{X_{22}X_{33}}$$

Sketch of Proof

- Calculate eigenvalues of Δ in terms of eigenvalues of the *deletion* and *link*:

$$\operatorname{del}_1 \Delta = \{F \in \Delta \mid 1 \notin F\},$$
$$\operatorname{link}_1 \Delta = \{F \in \Delta \mid 1 \notin F,\ F \cup \{1\} \in \Delta\}.$$

  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.

Sketch of Proof

- Calculate eigenvalues of Δ in terms of eigenvalues of the *deletion* and *link*:

$$\begin{aligned} \operatorname{del}_1 \Delta &= \{F \in \Delta \mid 1 \notin F\}, \\ \operatorname{link}_1 \Delta &= \{F \in \Delta \mid 1 \notin F,\ F \cup \{1\} \in \Delta\}. \end{aligned}$$

  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.

- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$

Sketch of Proof

- Calculate eigenvalues of Δ in terms of eigenvalues of the *deletion* and *link*:

$$\begin{aligned} \operatorname{del}_1 \Delta &= \{F \in \Delta \mid 1 \notin F\}, \\ \operatorname{link}_1 \Delta &= \{F \in \Delta \mid 1 \notin F,\ F \cup \{1\} \in \Delta\}. \end{aligned}$$

  - If Δ is shifted, then so are $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$.

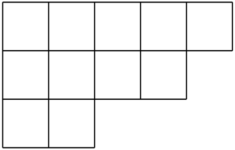
- Establish a recurrence for critical pairs of Δ in terms of those of $\operatorname{del}_1 \Delta$ and $\operatorname{link}_1 \Delta$
- "Here see ye two recurrences, and lo! they are the same."

Consequences of the Main Theorem

- Passing to the unweighted version (by setting $x_{i,j} = 1$ for all i, j) recovers the Duval–Reiner theorem.
- Special case $d = 1$: recovers known weighted spanning tree enumerators for threshold graphs (Remmel–Williamson 2002; JLM–Reiner 2003).
- A shifted complex is determined by its set of signatures, so we can "hear the shape of a shifted complex" from its Laplacian spectrum.

Ferrers Graphs

A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.



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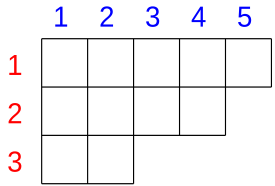

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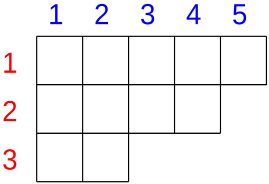

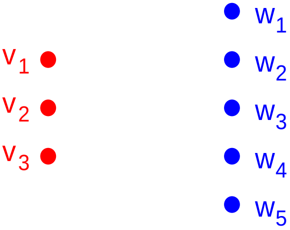

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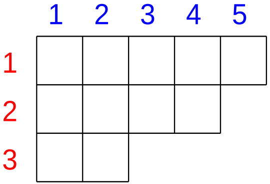

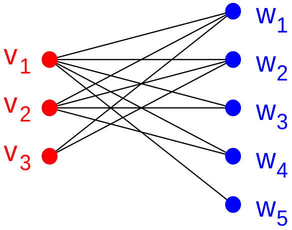

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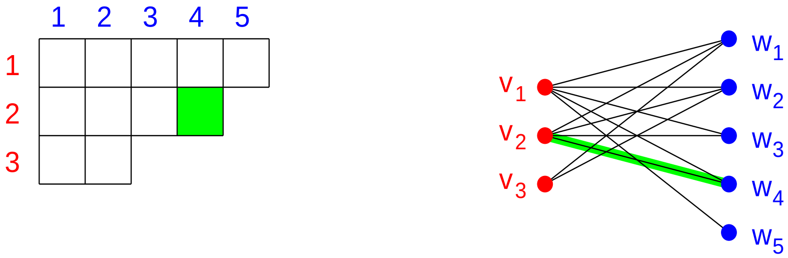

Ferrers Graphs

Ferrers graphs are bipartite analogues of threshold graphs.

- Degree-weighted spanning tree enumerator for Ferrers graphs: Ehrenborg and van Willigenburg (2004)
- Formula can also be derived from our finely weighted spanning tree enumerator for a threshold graph
- Higher-dimensional analogues?

Color-Shifted Complexes

Let Δ be a complex on $V = \bigcup_i V_i$, where

$$V_1 = \{v_{11}, \ldots, v_{1r_1}\}, \ \ldots, \ V_n = \{v_{n1}, \ldots, v_{nr_n}\}.$$

are disjoint vertex sets ("color classes").

Definition Δ is **color-shifted** if

- no face contains more than one vertex of the same color; and
- if $\{v_{1b_1}, \ldots, v_{nb_n}\} \in \Delta$ and $a_i \leq b_i$ for all i, then $\{v_{1a_1}, \ldots, v_{na_n}\} \in \Delta$.

Color-Shifted Complexes

- Color-shifted complexes generalize Ferrers graphs (Ehrenborg–van Willigenburg) and complete colorful complexes (Adin)
- Not in general Laplacian integral. . .
- . . . but they do seem to have nice degree-weighted spanning tree enumerators.

Matroid Complexes

Definition A pure simplicial complex Δ is a **matroid complex** if its facets form a matroid basis system:

- if F, G are facets and $i \in F \setminus G$,
- then there exists $j \in G \setminus F$ such that $F \setminus \{i\} \cup \{j\}$ is a facet.

Theorem [Kook–Reiner–Stanton 1999] Matroid complexes are Laplacian integral.

- Experimentally, degree-weighted spanning tree enumerators seem to have nice factorizations.