A Hopf Monoid of Set Families

Kevin Marshall and Jeremy L. Martin
University of Kansas

Hopfstravaganza
January 13, 2021

The Hopf Monoid **SetFam**

set family on E: $\mathcal{F} \subseteq 2^E$
grounded: $\varnothing \in \mathcal{F}$

restriction: $\mathcal{F}|_A = \{F \cap A : \ F \in \mathcal{F}\}$
contraction: $\mathcal{F}/_A = \{F \in \mathcal{F} : \ F \cap A = \varnothing\}$

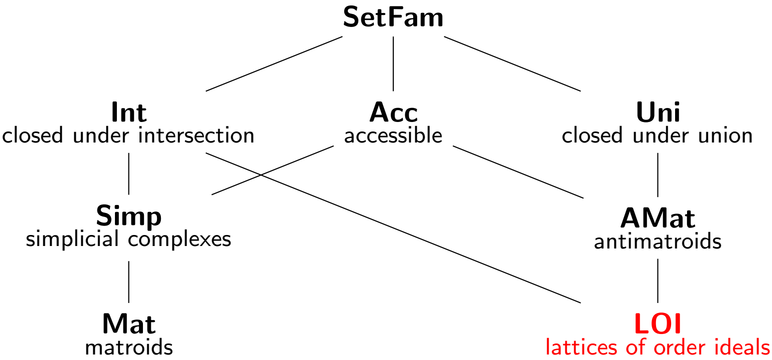
join: $\mathcal{F} * \mathcal{F}' = \{F \cup F' : \ F \in \mathcal{F}, \ F' \in \mathcal{F}'\}$

Proposition

The set species **SetFam** *of grounded set families carries the structure of a commutative Hopf monoid with multiplication given by join, and comultiplication*

$$\Delta_{A,B}(\mathcal{F}) = (\mathcal{F}|_A, \ \mathcal{F}/_A).$$

Submonoids of **SetFam**



- **Simp** is the maximal cocommutative submonoid ($\neq$ Benedetti–Hallam–Machacek)
- **Mat** is not the "classic" Hopf monoid of matroids (restriction is restriction but contraction is not contraction)

The Antipode in **LOI**

Let P be a finite poset on ground set E, so that $J(P) \in \mathbf{LOI}[E]$.
A **fracturing** of P is a disjoint sum of induced subposets of P.

Proposition

For every set composition $\Phi \vDash E$, there is a fracturing Q of P with

$$\mu_\Phi(\Delta_\Phi(J(P))) = J(Q).$$

Call a fracturing Q **good** if $X_P(Q) \neq \varnothing$, where

$$X_P(Q) := \{\Phi \vDash [n] : \ \mu_\Phi(\Delta_\Phi(J(P))) = J(Q)\}.$$

- $\Phi \in X_P(Q) \implies$ every component of Q is contained in a block of Φ
- The sets $X_P(Q)$ decompose the fan of the braid arrangement into subfans (not in general convex).

The Antipode in **LOI**

Theorem

Let P be a finite poset on ground set E, so that $J(P) \in \mathrm{LOI}[E]$. The antipode of $J(P)$ is then

$$S(J(P)) = \sum_{Q} (-1)^{u+k} J(Q)$$

where:

- *Q ranges over good fracturings of P;*
- *u = number of components of Q;*
- *k = number of "betrayed" elements of Q ($x \in E$ such that $x > y$ for some y occurring in an earlier block of Φ for any/all $\Phi \in X(Q)$)*

This formula is multiplicity- and cancellation-free.

Current Projects

- What is the antipode in **AMat**?
  - *Antimatroids are similar to lattices of order ideals in some respects — there is even an approximation of Birkhoff's theorem — but it is much more complicated.*
- What is the antipode in **Simp**?
  - *Probably inaccessible in general; complicated even for simplex skeletons.*

Current Projects

- What is the antipode in **AMat**?
  - *Antimatroids are similar to lattices of order ideals in some respects — there is even an approximation of Birkhoff's theorem — but it is much more complicated.*
- What is the antipode in **Simp**?
  - *Probably inaccessible in general; complicated even for simplex skeletons.*

Thank you!

jlmartin@ku.edu