Simplicial and Cellular Trees

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Spanning Trees

Let $G = (V, E)$ be a graph (connected, finite, not necessarily simple), with vertices $V = [n]$ and edges oriented arbitrarily.

Definition The **signed incidence matrix** $\partial = \partial(G)$ has rows and columns corresponding to vertices and edges of G, with entries

$$\partial_{v,e} = \begin{cases} +1 & \text{if } v = \text{head}(e) \\ -1 & \text{if } v = \text{tail}(e) \\ 0 & \text{if } v \notin e \text{ or } e \text{ is a loop} \end{cases}$$

Definition A **spanning tree** of G is a set of edges (or a subgraph) corresponding to a column basis of ∂.

$\mathscr{T}(G) =$ set of spanning trees of G; $\tau(G) = |\mathscr{T}(G)|$

The Laplacian Matrix

Definition Let G be a connected graph with vertices $[n] = \{1, \ldots, n\}$ and no loops. The **Laplacian** of G is the $n \times n$ matrix $L = \partial\partial^T = [\ell_{ij}]$:

$$\ell_{ij} \;=\; \begin{cases} \deg_G(i) & \text{if } i = j, \\ -(\text{number of edges between } i \text{ and } j) & \text{if } i \neq j. \end{cases}$$

- L is symmetric and positive semi-definite
- $\operatorname{rank} L = n - 1$
- $\ker L$ is spanned by the all-1's vector

The Matrix-Tree Theorem

Matrix-Tree Theorem [Kirchhoff 1847]

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n} .$$

(2) Let $1 \leq i \leq n$. Form the **reduced Laplacian** L_i by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det L_i .$$

The Matrix-Tree Theorem: Proof Sketch

Proof Sketch:

- Note that $L = \partial\partial^T$ and $L_i = \partial_i\partial_i^T$.
- Column bases of ∂ = spanning trees of G.
- Binet-Cauchy:

$$\det(\partial_i\partial_i^T) \;=\; \sum_{\substack{A\subseteq E(T)\\ |A|=n-1}} (\det \partial_A)^2 \;=\; \sum_{T\in\mathscr{T}(G)} (\pm 1)^2 \;=\; \tau(G).$$

Complete and Complete Bipartite Graphs

The **complete graph** K_n has n vertices, with every pair connected by one edge.

- Nonzero Laplacian spectrum: n^{n-1}
- $\tau(K_n) = n^{n-2}$ (Cayley's formula)

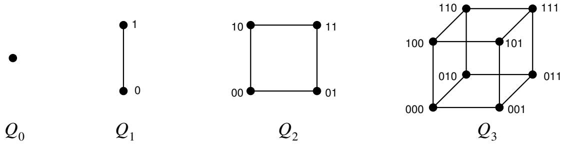
The **complete bipartite graph** $K_{p,q}$ has p red vertices and q blue vertices, with every red/blue pair connected by one edge.

- Nonzero Laplacian spectrum: $(p+q)^1 p^{q-1} q^{p-1}$
- $\tau(K_{p,q}) = p^{q-1} q^{p-1}$

Both these formulas can also be obtained bijectively [classical]

Hypercubes

The **hypercube graph** Q_n has 2^n vertices, labeled by strings of n bits (0's and 1's), with two vertices adjacent if they agree in all but one bit.



Theorem The eigenvalues of the Laplacian of Q_n are $0, 2, 4, \ldots, 2n$, with $2k$ having multiplicity $\binom{n}{k}$. Therefore,

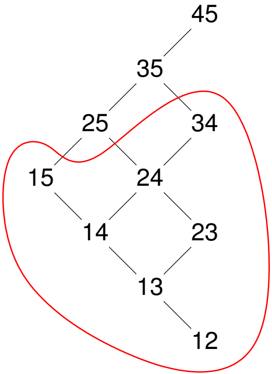
$$\tau(Q_n) = 2^{2^n - n - 1} \prod_{k=2}^{n} k^{\binom{n}{k}}.$$

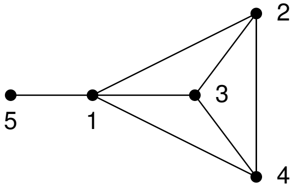
Combinatorial proof: [Bernardi '12]

Threshold Graphs

A graph G with vertex set $\{1, 2, \ldots, n\}$ is a **threshold graph** if, whenever ab is an edge, so is $a'b'$ for all $a' \leq a$ and $b' \leq b$.

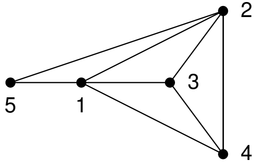
Equivalently, the edges of G form an order ideal under componentwise order.





Theorem [Merris '94] The eigenvalues of the Laplacian of a threshold graph G on vertices $[n]$ are the columns λ'_j of the partition $\lambda = \lambda(G)$ whose rows are the vertex degrees.

Corollary $\tau(G) = \lambda'_2 \lambda'_3 \cdots \lambda'_{n-1}$.



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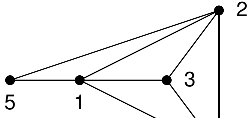

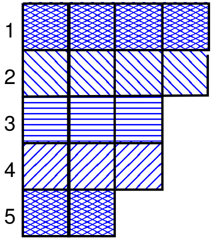

Vertex degrees: 4, 4, 3, 3, 2

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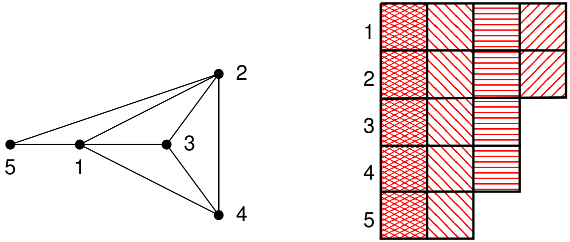

Laplacian eigenvalues: 5, 5, 4, 2, 0

Threshold Graphs

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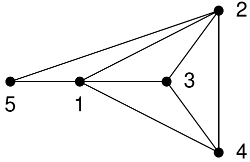
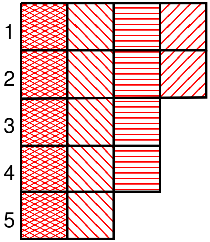




τ = 5 x 4 x 2 = 40 Laplacian eigenvalues: 5, 5, 4, 2, 0

Weighted Counting

$G = (V, E)$ graph; $\{x_e : \ e \in E\}$ commuting indeterminates

Weighted Laplacian $\hat{L} = [\hat{\ell}_{ij}]_{i,j \in V}$:

$$\hat{\ell}_{ij} \ = \ \begin{cases} \sum_{e \ni i} x_e & \text{if } i = j, \\ -\sum_{e = ij} x_e & \text{if } i \neq j. \end{cases}$$

Reduced Laplacian $\hat{\boldsymbol{L}}_{\boldsymbol{i}}$**:** pick a vertex i; delete i^{th} row and i^{th} column of $\hat{L}$

Weighted Matrix-Tree Theorem

$$\det L_i = \sum_{T \in \mathscr{T}(G)} \prod_{e \in T} x_e.$$

Weighted Counting

Combinatorial information about $\mathscr{T}(G)$ can be obtained by specializing edge weights x_e. Often, tree enumerators factor nicely.

- Complete graphs: $x_{ij} = x_i x_j$ gives *Cayley-Prüfer formula*

$$\sum_{T \in \mathscr{T}(K_n)} x_1^{\deg_T(1)} \cdots x_n^{\deg_T(n)} \;=\; x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}$$

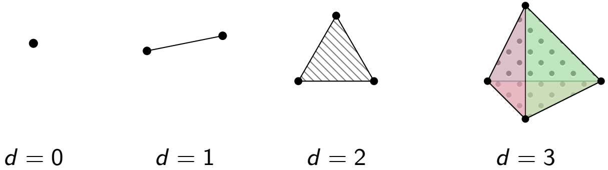
- Generalization to extended [Prüfer] graphs [Kelmans '92]
- Threshold graphs [Remmel–Williamson '02, JLM–Reiner '03]: factorization for bidegree generating function:

$$\sum_{T \in \mathscr{T}(G)} \prod_{e=i<j \in T} x_i y_j \;=\; x_1 y_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_{\min(i,r)} y_{\max(i,r)} \right)$$

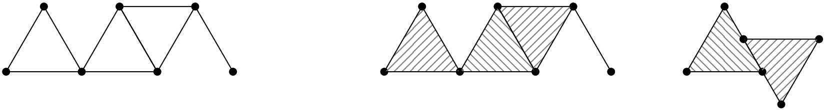
- Hypercubes: different weighting factors [JLM–Reiner '03]

Simplicial Complexes

A ***d*-simplex** is the convex hull of $d+1$ general points in $\mathbb{R}^{d+1}$.



A **simplicial complex** is a space built (properly) from simplices.



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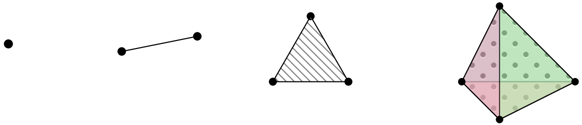
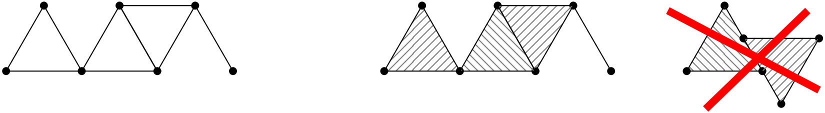


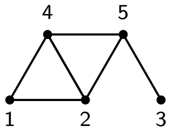
$d = 0$ $d = 1$ $d = 2$ $d = 3$

A **simplicial complex** is a space built (properly) from simplices.

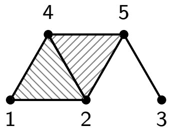


Simplicial Complexes

Combinatorially, a simplicial complex is a set family $\Delta \subseteq 2^{\{1,2,\ldots,n\}}$ such that if $\sigma \in \Delta$ and $\sigma' \subseteq \sigma$, then $\sigma' \in \Delta$.



$\Delta_1 = \langle 12, 14, 24, 24, 25, 35 \rangle$



$\Delta_2 = \langle 124, 245, 35 \rangle$

- **faces** or **simplices**: elements of Δ
- **dimension:** $\dim \sigma = |\sigma| - 1$
- **facet**: a maximal face
- **pure** complex: all facets have equal dimension
- ***k*-skeleton** $\Delta_{(k)} = \{\sigma \in \Delta : \dim \sigma \leq k\}$

Simplicial Boundary Maps and Homology

Let Δ be a simplicial complex on vertices $[n]$.
Write Δ_k for the set of k-dimensional faces.

The ***k^{th}* simplicial boundary matrix** of Δ is

$$\partial_k = \partial_k(\Delta) = [d_{\rho,\sigma}]_{\rho \in \Delta_{k-1},\, \sigma \in \Delta_k}$$

where

$$d_{\rho,\sigma} = \begin{cases} (-1)^j & \text{if } \sigma = \{v_0 < v_1 < \cdots < v_k\} \text{ and } \rho = \sigma \setminus v_j \\ 0 & \text{if } \rho \not\subseteq \sigma \end{cases}$$

Note: ∂_1 is the signed incidence matrix of the 1-skeleton of Δ.

Fact: $\ker \partial_k \supseteq \operatorname{im} \partial_{k+1}$ for all k.

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Definition For a ring R, the **homology groups of $\boldsymbol{\Delta}$ with coefficients in $\boldsymbol{R}$** are defined by

$$\tilde{H}_k(\Delta; R) \;=\; \ker(\partial_k; R) \;/\; \operatorname{im}(\partial_{k+1}; R).$$

(Default: $R = \mathbb{Z}$.)

Homology groups are topological invariants.

- $\tilde{H}_0(\Delta; R) = 0 \iff \Delta$ is connected
- $\tilde{H}_1(\Delta; R) = 0 \impliedby \Delta$ is simply connected
- Δ is contractible $\implies \tilde{H}_k(\Delta; R) = 0$ for all k, R

Simplicial Spanning Trees

Definition Let Δ^d be a pure simplicial complex of dimension d. A **spanning tree** (ST) is a complex Υ such that $\Delta_{(d-1)} \subseteq \Upsilon \subseteq \Delta$ and either of the following equivalent conditions hold:

1. The columns of $\partial_d(\Delta)$ corresponding to faces of Υ form a basis for its column space over $\mathbb{Q}$

   (i.e., Υ is a basis of the **simplicial matroid** of ∂_d).
2. $\tilde{H}_d(\Upsilon; \mathbb{Q}) = 0$ and $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$.
3. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ and $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is finite.

As before, let $\mathscr{T}(\Delta)$ denote the set of spanning trees of Δ.

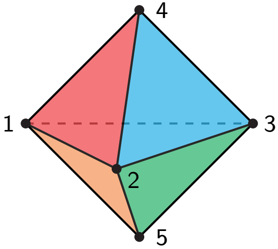
Note that we are **not** defining $\tau(\Delta)$ to be the cardinality of $\mathscr{T}(\Delta)$!

Examples of STs

- $\dim \Delta = 1$: $\mathscr{T}(\Delta)$ = usual graph-theoretic spanning trees
- $\dim \Delta = 0$: $\mathscr{T}(\Delta)$ = vertices of Δ
- If Δ is contractible: it has only one ST, namely itself.
  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- If Δ is a simplicial sphere: STs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any facet (maximal face)
  - Simplicial spheres are analogous to cycle graphs

Examples of STs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



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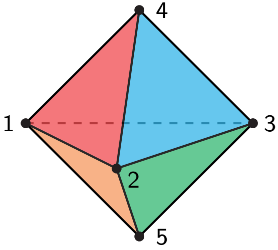
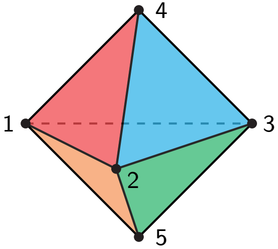


Solution: 15.

- Either remove triangle 123 and any other triangle (6 STs)...
- ...or one each "northern" and "southern" triangle (9 STs).

Examples of STs

Pop quiz: How many spanning trees does the equatorial bipyramid $\Delta = \langle 123, 124, 134, 234, 125, 135, 235 \rangle$ have?



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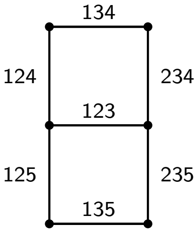

- Either remove triangle 123 and any other triangle (6 STs)...
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If Δ is a graph, then every spanning tree $\Upsilon \in \mathscr{T}(\Delta)$ is contractible, hence $\tilde{H}_0(\Upsilon; \mathbb{Z}) = 0$.

On the other hand, if $\dim \Delta = d \geq 2$ then $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ can be nontrivial.

Example $\Delta =$ complete 2-dimensional complex on 6 vertices; Υ = triangulation of $\mathbb{RP}^2$. Then

$$\tilde{H}_1(\Upsilon; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Geometrically: torsion suggests non-orientability.

Combinatorially: torsion affects the count of spanning trees.

Definition **Updown Laplacian matrix** of Δ in dimension $k-1$:

$$L^{\text{ud}}_{k-1}(\Delta) = \partial_k \partial_k^T.$$

- $L^{\text{ud}}_0(\Delta)$ is the usual graph Laplacian (rows/columns indexed by vertices).
- $L^{\text{ud}}_{k-1}(\Delta)$ is a symmetric square matrix with rows/columns indexed by $\rho, \pi \in \Delta_{k-1}$:

$$\ell_{\rho,\pi} = \begin{cases} \#\{\sigma \in \Delta_k \mid \sigma \supseteq \rho\} & \text{if } \rho = \pi, \\ \pm 1 & \text{if } \rho, \pi \text{ lie in a common } k\text{-face}, \\ 0 & \text{otherwise} \end{cases}$$

Reduced Laplacian $L_T(\Delta)$: pick a $(k-1)$-tree T and delete rows/columns corresponding to its facets

The Simplicial Matrix-Tree Theorem

Simplicial Matrix-Tree Theorem
(Bolker, Kalai, Adin, Duval–Klivans–JLM, ...)

The "number" of spanning trees of Δ^d is

$$\boxed{\tau_d(\Delta) \;\stackrel{\text{def}}{=}\; \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; c \det \hat{L}_T \;=\; \frac{c' \operatorname{pdet} L}{\tau_{d-1}(\Delta)}.}$$

- If $d = 1$ (graphs) then all summands are 1
- $\operatorname{pdet} M$ = product of nonzero eigenvalues (pseudodeterminant)
- Correction factors c, c' involve torsion homology; often trivial
- When do L and/or L_T have integer eigenvalues?

Kalai's Theorem

Complete d-dimensional complex on n vertices:

$$K_{n,d} = \{F \subseteq \{1,\ldots,n\} \;\mid\; \dim F \leq d\}$$

(In particular $K_{n,1} = K_n$.)

Theorem [Kalai '83]

$$\tau(K_{n,d}) \;=\; n^{\binom{n-2}{d}}.$$

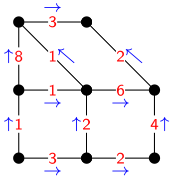
Better yet,

$$\sum_{\Upsilon \in \mathscr{T}(K)} |\tilde{H}_{d-1}(\Upsilon)|^2 \prod_{i=1}^{n} x_i^{\deg_\Upsilon(i)} = (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}.$$

Kalai's Theorem

- Kalai's theorem reduces to $\tau(K_n) = n^{n-2}$ when $d = 1$, and the weighted version reduces to Cayley-Prüfer.
- Bolker (1976): Observed that $n^{\binom{n-2}{d}}$ is an exact count of trees for small n, d, but fails for $n = 6$, $d = 2$.
  - The problem is torsion — $\mathbb{RP}^2$ requires six vertices to triangulate
- Adin (1992): Analogous formula for **complete colorful complexes**, generalizing $\tau(K_{n,m}) = n^{m-1}m^{n-1}$
- Duval–Klivans–JLM (2009): Enumeration for **shifted complexes** (I might get to this later)

A **[resistor] network** $N = (V, E, \mathbf{r})$ is a connected graph (V, E) together with positive **resistances** $\mathbf{r} = (r_e)_{e \in E}$.



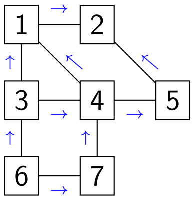
currents $\mathbf{i} = (i_{\vec{e}})_{e \in E}$

voltages $\mathbf{v} = (v_{\vec{e}})_{e \in E}$

Ohm's law	$i_e r_e = v_e \quad (\forall e \in E)$
Kirchhoff's current law	Net current out of every vertex is 0
Kirchhoff's voltage law	Net voltage gain around every cycle is 0

Every voltage comes from a **potential** $(p_x)_{x \in V}$ via $v_{\vec{xy}} = p_y - p_x$

Kirchhoff's Laws and the Incidence Matrix



∂	12	31	41	52	34	45	63	74	67
1	-1	1	1	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0
3	0	-1	0	0	-1	0	1	0	0
4	0	0	-1	0	1	-1	0	1	0
5	0	0	0	-1	0	1	0	0	0
6	0	0	0	0	0	0	-1	0	-1
7	0	0	0	0	0	0	0	-1	0
8	0	0	0	0	0	0	0	0	-1

KCL: $\mathbf{i} \in \ker \partial = \text{nullspace}(\partial)$

- Currents are **flows**

KVL: $\mathbf{v} \in (\ker \partial)^{\perp} = \text{rowspace}(\partial)$

- Voltages are **cuts**

Effective Resistance

Idea: Attach a **current generator**: edge $\mathbf{e} = \overrightarrow{xy}$ with current $i_{\mathbf{e}}$, then look for currents and voltages satisfying OL, KCL, KPL.

Dirichlet principle The state of the system is the unique minimizer of "total energy" $\sum_e v_e i_e$ subject to OL, KCL, KPL.

Rayleigh principle As far as the external world is concerned, the system is equivalent to a single edge $\mathbf{e}$ with resistance

$$R_{\mathbf{e}}^{\text{eff}} = R_{xy}^{\text{eff}} = \frac{p_y - p_x}{c_{\mathbf{e}}}.$$

(the **effective resistance** of $\mathbf{e}$).

Fact: If $(\mathbf{v}, \mathbf{i})$ obeys OL+KCL+KPL and minimize energy, then

$$R_{\mathbf{e}}^{\text{eff}} = v_{\mathbf{e}} / i_{\mathbf{e}}.$$

Effective Resistance and Tree Counting

Theorem [Thomassen 1990]
Let $N = (V, E, \mathbf{r})$ be a network and $e = xy \in E$.

- If $\mathbf{r} \equiv 1$, then

$$R_{xy}^{\text{eff}} = \frac{\tau(G/xy)}{\tau(G)} = \Pr[\text{random spanning tree contains } xy]$$

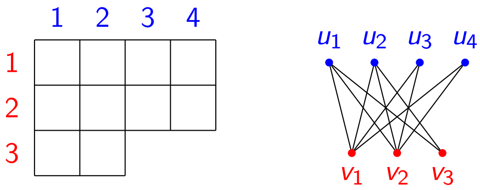
- Generalization for arbitrary resistances:

$$R_{xy}^{\text{eff}} = \frac{\hat{\tau}(G/xy)}{\hat{\tau}(G)} = \frac{\sum_{T \in \mathscr{T}(G/xy)} \prod_{e \in T} r_e^{-1}}{\sum_{T \in \mathscr{T}(G)} \prod_{e \in T} r_e^{-1}}.$$

Combinatorial application: weighted tree enumeration!

Application: Ferrers Graphs

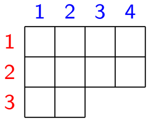
The **Ferrers graph** G_λ of a partition λ has vertices corresponding to the rows and columns of λ, and edges corresponding to squares.

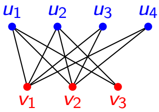


Here $\lambda = (4,4,2)$, $\lambda' = (3,3,2,2)$, $n = 3 = \ell(\lambda)$, $m = 4 = \ell(\lambda')$.
Define a degree-weighted tree enumerator

$$\hat{\tau}(G) = \sum_{T \in \mathscr{T}(G_\lambda)} \prod_{i=1}^{m} x_i^{\deg_T(u_i)} \prod_{j=1}^{n} y_j^{\deg_T(v_j)}$$

Application: Ferrers Graphs





Theorem (Ehrenborg and van Willigenburg, 2004):

$$\hat{\tau}(G_\lambda) = x_1 \cdots x_m \, y_1 \cdots y_n \prod_{i=2}^{n} (y_1 + \cdots + y_{\lambda_i}) \prod_{j=2}^{n} (x_1 + \cdots + x_{\lambda'_j})$$

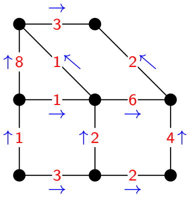
(Proof sketch: Find effective resistance of a corner of λ; induct.)
In the example above,

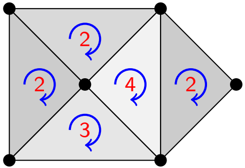
$$\hat{\tau}(G_\lambda) = x_1 x_2 x_3 x_4 \, y_1 y_2 y_3 \\ \times (y_1 + y_2 + y_3)(y_1 + y_2)^2 (x_1 + x_2 + x_3 + x_4)(x_1 + x_2)$$

and in particular $\tau(G_\lambda) = 3 \cdot 2^2 \cdot 4 \cdot 2$.

Simplicial network: pure complex Δ^d with resistances $(r_\varphi)_{\varphi\in\Phi}$ (Φ = facets of Δ)

$d = 1$





$d = 2$

Currents $\mathbf{i} = (i_\varphi)_{\varphi\in\Phi}$ Voltages $\mathbf{v} = (v_\varphi)_{\varphi\in\Phi}$

Ohm's law	$i_\varphi r_\varphi = v_\varphi$ for all $\varphi \in \Phi$
Kirchhoff's current law	$\mathbf{i} \in \ker(\partial_d)$
Kirchhoff's voltage law	$\mathbf{v} \in \ker(\partial_d)^\perp$

- Dirichlet, Rayleigh, R^{eff} have natural simplicial analogues.

Counting Simplicial Trees via Effective Resistance

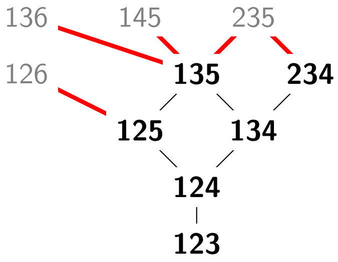
Theorem [Kook–Lee 2018]
Let $(\Delta, \mathbf{r})$ be a simplicial network and σ a current generator. Then:

$$R_\sigma^{\text{eff}} = \frac{\hat{\tau}(\Delta/\sigma)}{\hat{\tau}(\Delta)} = \frac{\displaystyle\sum_{T \in \mathscr{T}(\Delta/\sigma)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\varphi \in T} r_\varphi^{-1}}{\displaystyle\sum_{T \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(T, \mathbb{Z})|^2 \prod_{\varphi \in T} r_\varphi^{-1}}.$$

- Generalizes Thomassen's theorem for R^{eff} in graphs
- Δ/σ = quotient complex (not simplicial, but close enough)
- Application: count trees by induction on facets (a la Ehrenborg–van Willigenburg)

Shifted Complexes

A (pure) simplicial complex Δ on vertices $\{1, \ldots, n\}$ is **shifted** if any vertex of a face may be replaced with a smaller vertex. Equivalently, the facets of Δ form an order ideal in *Gale order* or *componentwise order* (best explained by a picture)



$\Delta = \langle 135, 234 \rangle_{\text{Gale}}$

Facets

Nonfaces

Critical pairs

Shifted complexes are **nice:** shellable, good h-vectors, arise in algebra (Borel-fixed ideals), generalize threshold graphs

Duval–Reiner '02: Let λ_i = number of max-dim faces containing vertex i. Then eigenvalues of $L(\Delta)$ = column lengths of λ.

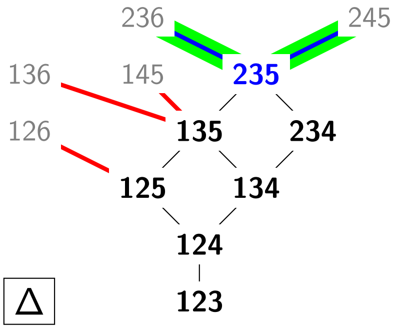
(Generalizes Merris's Theorem — one-dimensional shifted complexes are just threshold graphs.)

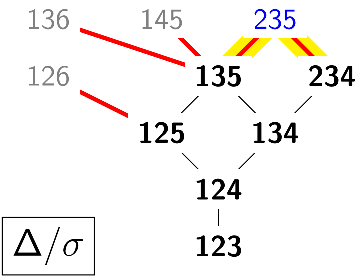
Duval–Klivans–JLM '09: recursion for $\hat{\tau}(\Delta)$ via the shifted complexes $\langle \varphi \in \Delta \mid 1 \in \varphi \rangle$ and $\langle \varphi \in \Delta \mid 1 \notin \varphi \rangle$.
Here $\hat{\tau}(\Delta)$ is the finely weighted degree enumerator

$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |H_{d-1}(\Upsilon; \mathbb{Z})|^2 \prod_{\substack{\text{facets} \\ \{v_0 < \cdots < v_d\}}} x_{0,v_0} \cdots x_{d,v_d}$$

Punchline: Critical pairs P correspond to factors f_P of $\hat{\tau}(\Delta)$.

Shifted Complexes





$$R^{\text{eff}}(\sigma) = \frac{\hat{\tau}(\Delta/\sigma)}{\hat{\tau}(\Delta)} = \frac{\prod_{\text{yellow } P} f_P}{\prod_{\text{green } P} f_P}$$

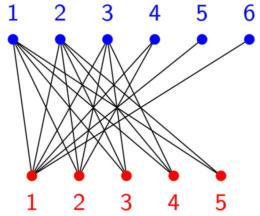
Color-Shifted Complexes

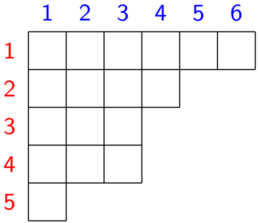
A simplicial complex Δ^d is **color-shifted** [Babson–Novik '06] if:

- $V(\Delta) = V_1 \cup \cdots \cup V_{d+1}$, where $V_q = \{v_{q1}, \ldots, v_{q\ell_q}\}$
- Each facet contains exactly one vertex of each color
- A vertex may be replaced with a smaller vertex of same color

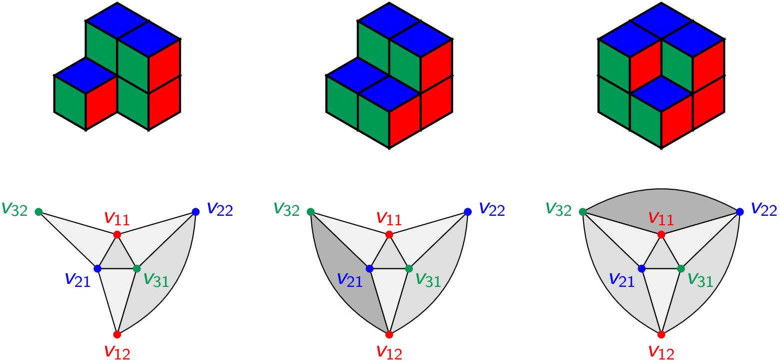
(Equivalently, facets are an order ideal in $V_1 \times \cdots \times V_q$.)

- A 1-dimensional color-shifted complex is just a Ferrers graph.





Color-Shifted Complexes



Trees in Color-Shifted Complexes

Vertex-weighted spanning tree enumerators:

$$\hat{\tau}(\Delta) = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{\varphi \in \Upsilon} \prod_{v_{qj} \in \varphi} x_{qj}$$
$$= \sum_{\Upsilon \in \mathscr{T}(\Delta)} |H_{d-1}(\Upsilon;\mathbb{Z})|^2 \prod_{q,j} x_{qj}^{\deg_\Upsilon(v_{qj})}$$

Proposition [Duval–Kook–Lee–JLM 2021$^+$]
Let Δ^d be color-shifted and $\sigma = v_{1,k_1} \dots v_{d+1,k_{d+1}}$ a minimal nonfacet. Then

$$R^{\text{eff}}(\sigma) = \frac{\hat{\tau}(\Delta + \sigma)}{\tau(\Delta)} = \prod_{q=1}^{d+1} \frac{x_{q,1} + \cdots + x_{q,k_q}}{x_{q,1} + \cdots + x_{q,k_q - 1}}.$$

Trees in Color-Shifted Complexes

Theorem [Duval–Kook–Lee–JLM 2022$^+$]

$$\hat{\tau}(\Delta) = \prod_{q,i} x_{q,i}^{e(q,i)} \prod_{\substack{\rho \in \Delta \\ \dim \rho = d-1}} (x_{m(\rho),1} + \cdots + x_{m(\rho),k(\rho)})$$

where

$$\begin{aligned} e(q,i) &= \#\{\sigma \in \Delta_d \mid v_{q,i} \in \sigma \text{ and } v_{q',1} \in \sigma \text{ for some } q' \neq q\} \\ m(\rho) &= \text{unique color missing from } \rho \\ k(\rho) &= \max\{j \mid \rho \cup v_{m(\rho),j} \in \Delta\} \end{aligned}$$

- Special case $d = 1$ is Ehrenborg–van Willigenburg
- Previously conjectured by Aalipour and Duval [unpublished]
- Result seems inaccessible without effective resistance

Thank you!

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