A positivity phenomenon in Elser's Gaussian-cluster percolation model

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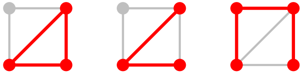
Nuclei and Elser Numbers

Definition

A **nucleus** of a connected graph G is a connected subgraph $N \subseteq G$ such that $V(N)$ is a vertex cover.



Definition

Let $\mathcal{N}(G)$ denote the set of all nuclei. The $\boldsymbol{k}^{\mathbf{th}}$ **Elser number** of G is

$$\mathrm{els}_k(G) = (-1)^{|V(G)|+1} \sum_{N \in \mathcal{N}(G)} (-1)^{|E(N)|} |V(N)|^k.$$

Example $\mathrm{els}_k(K_2) = 2^k - 2$

Idea: Interpret Elser numbers as Euler characteristics.

Motivation: Percolation Theory

- **Percolation theory** models a physical medium by an Erdös-Rényi random subgraph Γ of $\mathbb{Z}^d$ or some other periodic lattice [H. Kesten, Notices AMS 2006]
- Typically, different values of d have to be studied separately (think the Drunkard's Walk)
- V. Elser [J. Phys. A 1984] proposed a random *geometric* graph model in which d can be treated as a parameter
- The numbers $\text{els}_k(G)$ arise in a generating function for connected components of Γ.
- Elser proved that $\text{els}_1(G) = 0$ for all G and conjectured (based on experimentation) that $\text{els}_k(G) \geq 0$ for all $k \geq 2$.

The Main Theorem

Theorem [D-B, H, L, M, N, V, W 2019^+]
Let G be a connected graph with at least two vertices. Then:

1. $\text{els}_0(G) \leq 0$.
2. $\text{els}_1(G) = 0$.
3. $\text{els}_k(G) \geq 0$ for all $k \geq 2$ (Elser's conjecture).

More specifically:

4. If G is 2-connected, then $\text{els}_0(G) < 0$ and $\text{els}_k(G) > 0$ for all $k \geq 2$.
5. Otherwise, $\text{els}_k(G) \neq 0$ if and only if $k \geq \ell$, where $\ell \geq 2$ is the number of *leaf blocks* (2-connected components of G that contain exactly one cut-vertex).

Also:

6. (Monotonicity) If $e \in E(G)$ is neither a loop or cut-edge, then

$$\text{els}_k(G) \geq \text{els}_k(G/e) + \text{els}_k(G \setminus e)$$

with equality for $k = 0$.

Nucleus Complexes

Nuclei are stable under adding edges. Therefore:

Definition

The **nucleus complex** of G is the simplicial complex on $E(G)$ given by

$$\Delta^G = \{E(\overline{N}) : \ N \in \mathcal{N}(G)\}.$$

For $U \subseteq V(G)$, the ***U*-nucleus complex** is the subcomplex

$$\Delta^G_U = \{E(\overline{N}) : \ N \in \mathcal{N}(G), \ V(N) \supseteq U\}.$$

Annoying Special Case: We have to define $\Delta_\emptyset^{cK_2}$ as a non-simplicial Δ-complex, since not all nuclei of cK_2 are determined by their edge sets.

Elser Numbers and Euler Characteristics

What do the simplicial complexes Δ_U^G look like?

- $\Delta_\emptyset^G = \Delta^G$
- $\Delta_{V(G)}^G$ = matroid complex of cographic matroid
- In general, U-nucleus complexes are not pure or (nonpure) shellable

Proposition

$$\begin{aligned}\mathrm{els}_k(G) &= (-1)^{|E(G)|+|V(G)|+1} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|) \sum_{\substack{N \in \mathcal{N}(G): \\ U \subseteq V(N)}} (-1)^{|E(\overline{N})|} \\ &= (-1)^{|E(G)|+|V(G)|} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_U^G) \qquad (1)\end{aligned}$$

where $\mathrm{Sur}(a, b) = b! \cdot \mathrm{Stir2}(a, b)$ = number of surjections $[a] \to [b]$.

U-Nucleus Complexes

Let $\mathrm{Dep}(G)$ be the **deparallelization** of G: identify all edges in the same parallel class. (Note that $\mathrm{Dep}(G)$ can have loops.)

Proposition Let G be a graph and $U \subseteq V(G)$.

1. If G has a loop ℓ, then ℓ is a cone point of Δ_U^G, so $\tilde{\chi}(\Delta_U^G) = 0$.
2. Let $D = \mathrm{Dep}(G)$. Then $\tilde{\chi}(\Delta_U^D) = (-1)^{|E(G)|-|E(D)|}\tilde{\chi}(\Delta_U^G)$.
3. Suppose $G \neq K_2$ has a cut-edge e. Then:
   - If e is a leaf edge with leaf x and $x \notin U$, then Δ_U^G is a cone.
   - Otherwise, $\Delta_U^G = \Delta_{U/e}^{G/e}$.

In terms of Elser numbers:

1′. If G has a loop, then $\mathrm{els}_k(G) = 0$ for all k.

2′. For all G and k, $\mathrm{els}_k(G) = \mathrm{els}_k(\mathrm{Dep}(G))$.

Elser Numbers for Trees

Corollary Let T be a tree with $n \geq 3$ vertices. Let L be the set of leaf vertices. Then:

$$\tilde{\chi}(\Delta_U^T) = \begin{cases} 1 - |U| & \text{if } T = K_2, \\ 0 & \text{if } T \neq K_2 \text{ and } L \not\subseteq U, \\ -1 & \text{if } T \neq K_2 \text{ and } L \subseteq U. \end{cases} \tag{2}$$

Corollary For all $k \geq 1$,

$$\text{els}_k(T) = \sum_{i=0}^{n-|L|} \binom{n-|L|}{i} \text{Sur}(k, |L| + i)$$

In particular, Elser's conjecture is true for trees: $\text{els}_k(T) \geq 0$ for $k \geq 1$ (and > 0 iff $k \geq |L|$).

Deletion/Contraction for Nucleus Complexes

Theorem Let G be a connected graph with $|V(G)| \geq 2$, let $e \in E(G)$ be neither a loop or cut-edge, and let $U \subseteq V(G)$. Then

$$\tilde{\chi}(\Delta_U^G) = \tilde{\chi}(\Delta_{U/e}^{G/e}) - \tilde{\chi}(\Delta_U^{G\setminus e}).$$

Proof sketch: It's a lot like proving a Tutte polynomial identity.

Define a bijection $\psi : 2^{E(G)} \rightarrow 2^{E(G\setminus e)} \cup\!\!\!\cdot\; 2^{E(G/e)}$ by

$$\psi(A) = \begin{cases} A \setminus e \;\subseteq\; E(G \setminus e) & \text{if } e \in A, \\ A \;\subseteq\; E(G/e) & \text{if } e \notin A. \end{cases}$$

Then crank out the recurrence, keeping track of vertex sets and treating the case $G = K_2$ separately.

Proof of Elser's Conjecture

Idea Apply the deletion/contraction recurrence repeatedly until it bottoms out. The result will be an expression

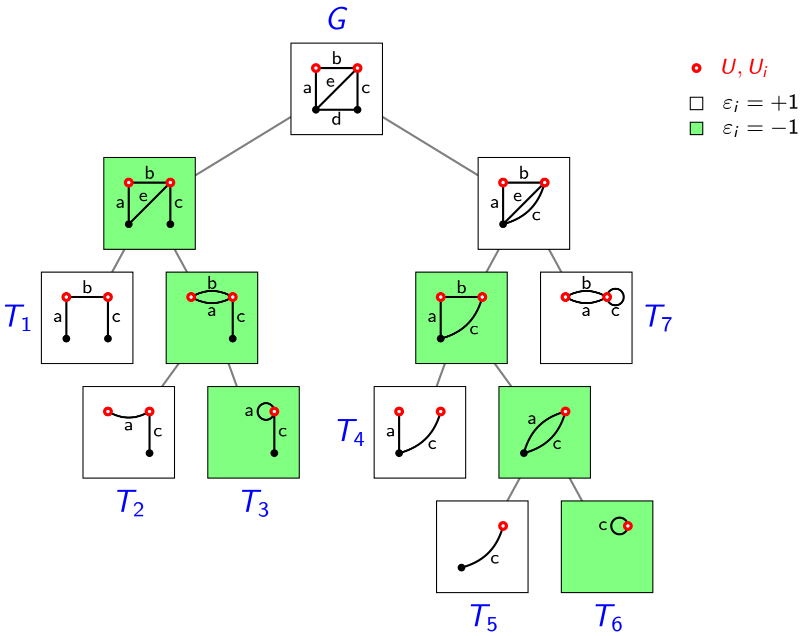
$$\tilde{\chi}(\Delta_U^G) = \sum_{i=1}^{s} \varepsilon_i \; \tilde{\chi}(\Delta_{U[T_i]}^{T_i}) \tag{3}$$

where

- $T_1, \ldots, T_s$ are tree minors of G;
- $U[T_i]$ = image of U in T_i;
- $\varepsilon_i \in \{\pm 1\}$.

The list $T_1, \ldots, T_s$ is not an invariant of G, but depends on the choices of edge to delete and contract at each stage of the recurrence.

The calculation is recorded by a thing we call a **restricted deletion/contraction tree** (RDCT).


G
b
a
e
c
d
T_1
T_2
T_3
T_4
T_5
T_6
T_7
U, U_i
$\varepsilon_i = +1$
$\varepsilon_i = -1$

Proof of Elser's Conjecture

Observation

$\varepsilon_i =$ number of edges deleted $= \big(|E(G)| - |E(T_i)|\big) - \big(|V(G)| - |V(T_i)|\big)$

Therefore, formula (3) becomes

$$(-1)^{|E(G)|+|V(G)|}\tilde{\chi}(\Delta_U^G) \;=\; -\sum_{i=1}^{s} \tilde{\chi}(\Delta_{U[T_i]}^{T_i}) \quad \begin{cases} \leq 0 & \text{if } |U| = 0 \\ = 0 & \text{if } |U| = 1 \\ \geq 0 & \text{if } |U| \geq 1 \end{cases}$$

and Elser's conjecture follows from the calculation of Elser numbers for trees (2) and the relationship between Elser numbers and $\tilde{\chi}(\Delta_U^G)$ (1).

Question

When are the inequalities ≤ 0 and ≥ 0 strict?

Pinning Down The Signs

Proposition

Let $\mathscr{B}$ be any RDCT for G, with leaves $T_1, \ldots, T_s$. Then:

1. $\mathrm{els}_0(G) = -\#\{i : \ T_i \cong K_2\}$.
2. For $k \geq 2$, the following are equivalent:
   - $\mathrm{els}_k(G) > 0$;
   - for some $U \subseteq V(G)$ we have $|U| \leq k$ and $\tilde{\chi}(\Delta_U^G) \neq 0$;
   - for some $i \in [s]$ we have $|L(T_i)| \leq k$.

Problem

This result depends on the choice of $\mathscr{B}$ — we would like a description in terms of G itself.

Question

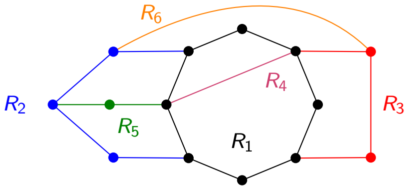
Which tree minors show up as leaves of $\mathscr{B}$ for *some* $\mathscr{B}$?

Ear Decompositions

Definition

An **ear decomposition** of G is a list of subgraphs $R_1, \ldots, R_m$ such that

1. $E(G) = E(R_1) \cup \cdots \cup E(R_m)$;
2. R_1 is a cycle; and
3. for each $i > 1$, the graph R_i is a path that meets $R_1 \cup \cdots \cup R_{i-1}$ only at its endpoints.



Fact G has an ear decomposition if and only if it is 2-connected (i.e., has no cut-vertex).

Criteria for Positivity of Elser Numbers

Theorem [D-B–H–L–M–N–V–W; F. Petrov]

Let G be a 2-connected graph and let $T \subseteq G$ be a spanning tree.

Then G has an ear decomposition $R_1 \cup \cdots \cup R_m$ such that $|E(R_i) \setminus T| = 1$ for every i.

(Proof: constructive algorithm)

Corollary Let G be 2-connected.

Then *every* tree minor of G can be realized without contracting a cut-edge or deleting a loop, hence appears as a leaf of some RDCT.

Consequently, $\text{els}_k(G) > 0$ for all $k \geq 2$.

Proposition
More generally, $\text{els}_k(G) \neq 0$ if and only if G has at most k leaf blocks.

Future Directions

Open Question

To what extent do Elser numbers depend on the matroid of G? (They are *not* matroid invariants.)

Conjecture

Let G be a connected graph and $U \subseteq V(G)$. Then the reduced Betti number $\tilde{\beta}_k(\Delta_U^G)$ is nonzero only if

(i) $U = \emptyset$ and $k = |E(G)| - |V(G)| - 1$, or

(ii) $|U| \geq 2$ and $k = |E(G)| - |V(G)|$.

- Can be reduced to the case of 2-connected graphs
- True when U is a vertex cover (using Jakob Jonsson's theory of *pseudo-independence complexes*)
- Verified computationally for $|V(G)| \leq 6$
- What are the Betti numbers??

Open Question

What significance does our result have for Elser's percolation model???

Thank You!

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