Cuts and Flows in Cell Complexes

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The Incidence Matrix

$G = (V, E)$: connected, loopless graph

Orient each edge e by labeling its endpoints head and tail.

Signed incidence matrix $\partial = [\partial_{ve}]_{v \in V,\ e \in E}$:

$$\partial_{ve} = \begin{cases} 1 & \text{if } v = \text{head}(e) \\ -1 & \text{if } v = \text{tail}(e) \\ 0 & \text{otherwise} \end{cases}$$

Cut and Flow Spaces

Definition

The cut space and flow space of G are

$$\mathrm{Cut}(G) = \mathrm{im}\,\partial^* \subseteq \mathbb{R}^E, \qquad \mathrm{Flow}(G) = \ker \partial \subseteq \mathbb{R}^E.$$

- Flow vectors $\phi = (\phi_e)_{e \in E}$ are defined by the condition

$$\sum_{e:\ v=\mathrm{head}(e)} \phi_e - \sum_{e:\ v=\mathrm{tail}(e)} \phi_e = 0 \qquad \forall v \in V.$$

- Typical cut vector χ: fix a partition $V = X \dot\cup Y$ and define

$$\chi_e = \begin{cases} 1 & \text{if head}(e) \in X \text{ and tail}(e) \in Y \\ -1 & \text{if head}(e) \in Y \text{ and tail}(e) \in X \\ 0 & \text{otherwise} \end{cases}$$

Cut and Flow Spaces and Lattices

- The flow and cut spaces are orthogonal complements in $\mathbb{R}^E$. $\dim \operatorname{Flow}(G) = |E| - |V| + 1$ and $\dim \operatorname{Cut}(G) = |V| - 1$.

Fix a spanning tree T.

- For each edge $e \notin T$, there is a unique cycle in $T \cup e$. The characteristic vectors of all such cycles form a basis for $\operatorname{Flow}(G)$.
- For each edge $e \in T$, the graph with edges $T \setminus e$ has two components. The corresponding cut vectors form a basis for $\operatorname{Cut}(G)$.
- These bases are in fact $\mathbb{Z}$-module bases for the cut lattice $\mathcal{C}(G) = \operatorname{Cut}(G) \cap \mathbb{Z}^E$ and the flow lattice $\mathcal{F}(G) = \operatorname{Flow}(G) \cap \mathbb{Z}^E$.

The Laplacian and the Critical Group

Laplacian matrix: $L = \partial\partial^* = [\ell_{xy}]_{x,y\in V}$

$$\ell_{xy} = \begin{cases} |\{\text{edges incident to } x\}| & \text{if } x = y \\ -|\{\text{edges joining } x, y\}| & \text{if } x \neq y \end{cases}$$

Definition

The critical group $K(G)$ is the torsion summand of $\operatorname{coker} L := \mathbb{Z}^n / \operatorname{im} L$.

Alternately, if $\tilde{L}_i$ is the *reduced Laplacian* obtained from L by deleting the i^{th} row and column, then $K(G) = \operatorname{coker} \tilde{L}$.

By the Matrix-Tree Theorem, $|K(G)|$ is the number of spanning trees of G.

Cuts, Flows and The Critical Group

The dual of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ is $\mathcal{L}^\sharp = \{w \in \mathcal{L} \otimes \mathbb{R} \mid v \cdot w \in \mathbb{Z} \;\; \forall v \in \mathcal{L}\}$.

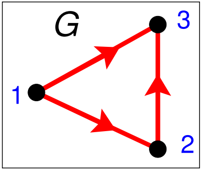
Theorem (Bacher, de la Harpe, Nagnibeda)

For every graph G, there are isomorphisms

$$K(G) \;\cong\; \mathcal{F}^\sharp/\mathcal{F} \;\cong\; \mathcal{C}^\sharp/\mathcal{C} \;\cong\; \mathbb{Z}^E/(\mathcal{C} \oplus \mathcal{F}).$$

- Chip-firing game: elements of critical group correspond to long-term behaviors of the chip-firing game/sandpile model
- Tutte polynomial / G-parking functions
- Graph : Riemann surface :: Critical group : Picard group (BdlHN, Baker–Norine)

Example: $G = K_3$



$$\partial = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Flow lattice

$$\mathcal{F} = \ker \partial = \langle (1, -1, 1) \rangle$$

$$\mathcal{F}^\sharp = \left\langle \left(\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\right) \right\rangle$$

Cut lattice

$$\mathcal{C} = \operatorname{im} \partial^* = \langle (1, 0, -1),\ (0, 1, 1) \rangle$$

$$\mathcal{C}^\sharp = \left\langle \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right),\ \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right) \right\rangle$$

Note: $\mathcal{F}^\sharp/\mathcal{F} = \mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}^3/(\mathcal{C} \oplus \mathcal{F}) = K(G) = \mathbb{Z}/3\mathbb{Z}$

Cell Complexes

Cell complexes are higher-dimensional generalizations of graphs (like simplicial complexes, but even more general).

Examples: graphs, simplicial complexes, polytopes, polyhedral fans, . . .

Rough definition: A cell complex X consists of cells (homeomorphic copies of $\mathbb{R}^k$ for various k) together with attaching maps

$$\partial_k(X) : C_k(X) \to C_{k-1}(X)$$

where $C_k(X)$ = free $\mathbb{Z}$-module generated by k-dimensional cells. (Note: $\partial_k \partial_{k+1} = 0$ for all k.) The integer $\partial_k(X)_{\rho,\sigma}$ specifies the multiplicity with which the k-cell σ is attached to the $(k-1)$-cell ρ.

Notation: $X_{(k)}$ = k-skeleton of X (union of all cells of dimension $\leq k$)

The Cellular Critical Group

Definition

The critical group of X^d is $K(X) = \ker \partial_{d-1} / \operatorname{im} \partial_d \partial_d^*$.

Fact: $K(X)$ is finite abelian of order $\tau(X)$, and can also be expressed in terms of the reduced Laplacian [DKM '11]

Questions:

- Can we interpret $K(X)$ in terms of cuts and flows?
- Is there a cellular chip-firing game for which elements of $K(X)$ correspond to critical states?
- Further discrete analogues of graphical Riemann-Roch?

Cellular Cuts and Flows

Definition

The cut and flow spaces of X are $\mathrm{Cut}(X) = \mathrm{im}\,\partial^*$ and $\mathrm{Flow}(X) = \ker \partial$ (considered as vector spaces over $\mathbb{R}$). The cut and flow lattices are $\mathcal{C}(X) = \mathrm{im}\,\partial^*$ and $\mathcal{F}(X) = \ker \partial$ (considered as $\mathbb{Z}$-modules).

Theorem (DKM)

Fix a cellular spanning tree $Y \subset X$.

1. *There are natural bases of $\mathrm{Cut}(X)$ and $\mathrm{Flow}(X)$ indexed by the d-cells contained / not contained in Y.*
2. *The basis element corresponding to a d-cell σ is supported on the fundamental cocircuit / circuit of σ w.r.t. Y, and the coefficients are the cardinalities of certain (relative) homology groups.*
3. *Under certain conditions on $\tilde{H}_{d-1}(Y)$, these are $\mathbb{Z}$-module bases for $\mathcal{C}$ and $\mathcal{F}$.*

Cellular Cuts and Flows

Question

Do the Bacher-de la Harpe-Nagnibeda isomorphisms

$$K(G) \;\cong\; \mathcal{F}^\sharp/\mathcal{F} \;\cong\; \mathcal{C}^\sharp/\mathcal{C} \;\cong\; \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F})$$

still hold if the graph G is replaced with an arbitrary cell complex?

Answer: Not quite.

Cellular Cuts and Flows

Example: $X = \mathbb{R}P^2$: cell complex with one vertex, one edge, and one 2-cell, and cellular chain complex

$$C_2 = \mathbb{Z} \xrightarrow{\partial_2 = [2]} C_1 = \mathbb{Z} \xrightarrow{[\partial_1 = 0]} C_0 = \mathbb{Z}$$

$$\mathcal{C} = \operatorname{im} \partial_2^* = 2\mathbb{Z} \qquad \mathcal{F} = \ker \partial_2 = 0 \quad \mathbb{Z}/(\mathcal{C} \oplus \mathcal{F}) = \mathbb{Z}/2\mathbb{Z}$$

$$\mathcal{C}^\sharp = \tfrac{1}{2}\mathbb{Z} \qquad \mathcal{F}^\sharp/\mathcal{F} = 0$$

$$\mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}/4\mathbb{Z}$$

$$K(X) = \ker \partial_1 / \operatorname{im} \partial_2\partial_2^* = \mathbb{Z}/4\mathbb{Z}$$

The problem is torsion (which doesn't show up in graphs). Note:
$\tilde{H}_2(X) = 0$; $\tilde{H}_1(X) = \mathbb{Z}/2\mathbb{Z}$; $\tilde{H}_0(X) = 0$.
(For a connected graph G: $\tilde{H}_1(G) = \mathbb{Z}^{|E|-|V|+1}$, $\tilde{H}_0(G) = 0$.)

The Main Theorem

Theorem (DKM)

For any cell complex X, there are short exact sequences

$$0 \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K(X) \cong \mathcal{C}^\sharp/\mathcal{C} \to \mathbf{T}(\tilde{H}_{d-1}(X)) \to 0$$

and

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(X)) \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to K^*(X) \cong \mathcal{F}^\sharp/\mathcal{F} \to 0.$$

Brief algebra review: "$0 \to A \to B \to C \to 0$ is a short exact sequence" is equivalent to "$C \cong B/A$."
$\mathbf{T}(A)$ means the torsion summand of A.
Methods: Lots of homological algebra

Cellular Cuts and Flows

What's this thing called $K^*(X)$?
Cocritical group: First, construct an "acyclization" Ω of X by adjoining $(d+1)$-cells so as to eliminate all d-homology.
Then, define $K^*(X) = C_{d+1}(\Omega;\mathbb{Z})/\operatorname{im}\partial^*_{d+1}\partial_{d+1} = \operatorname{coker} L^{\mathrm{du}}_{d+1}(\Omega)$.
PROBABLY NEED AN EXAMPLE.
(Compare: $K(X) = \ker\partial_{d-1}/\operatorname{im} L^{\mathrm{ud}}_{d-1}$.)

Ongoing/Future Work

Chip-firing/sandpiles/Riemann-Roch theory in higher dimension (connections to Baker-Norine; combinatorial commutative algebra connection (Hopkins/Perkinson/Wilmes, Dochtermann–Sanyal, Mohammadi–Shokrieh, etc.)

Max-flow/min-cut