On the Chromatic Symmetric Function of a Tree

Jeremy Martin and Jennifer Wagner
University of Kansas

jmartin@math.ku.edu
http://math.ku.edu/∼jmartin/pubs.html

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Warning! Attention! ¡Cuidado!

Our FPSAC '06 extended abstract has been superseded by stronger results.

Please refer to the article "On the Chromatic Symmetric Function of a Tree" by Jeremy Martin, Matthew Morin, and Jennifer Wagner (in preparation).

Chromatic Symmetric Functions

G = finite simple graph
$V(G)$ = vertices
$E(G)$ = edges
$n = \#V(G)$
$\underline{x} = \{x_1, x_2, \dots\}$ = commuting indeterminates

Coloring of G: a function $\kappa : V(G) \to \mathbb{N}$ such that

$$vw \in E(G) \implies \kappa(v) \neq \kappa(w)$$

Chromatic symmetric function of G:

$$\mathbf{X}_G = \mathbf{X}_G(\underline{x}) = \sum_{\text{colorings } \kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

(Stanley 1995)

- Symmetric in $x_1, x_2, \dots$
- Homogeneous of degree n
- Stronger invariant than the chromatic polynomial

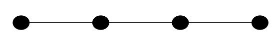
Examples

- $G = K_n$ (complete graph on n vertices)

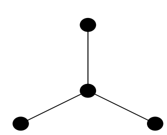
  $\mathbf{X}_G = e_n = e_n(\underline{x})$

- $G = \overline{K_n}$ (n vertices, no edges)

  $\mathbf{X}_G = p_{1^n} = (x_1 + x_2 + \cdots)^n$

- $G = P_3$ 

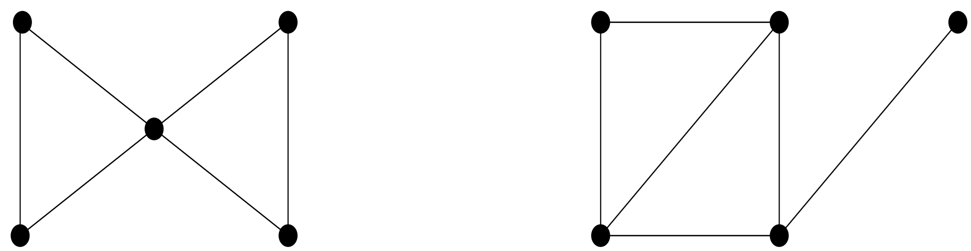
  $\mathbf{X}_G = 24m_{1111} + 6m_{211} + 2m_{22}$

- $G = S_3$ 

  $\mathbf{X}_G = 24m_{1111} + 6m_{211} + m_{31}$

X_G is not a complete invariant

Example (Stanley): The following two nonisomorphic graphs have the same chromatic symmetric function:

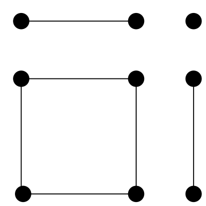


Open Question: If T is a tree, does $\mathbf{X}_T$ determine T up to isomorphism?

- Yes for $n \leq 23$ (Tan 2006)
- Yes for certain special families of graphs (spiders, some caterpillars)

Coefficients of $\mathbf{X}_G$

For $A \subseteq E(G)$, let $\lambda(A)$ be the partition of n whose parts are the sizes of the components of A.

$A =$  $\lambda(A) = (4, 2, 2, 1)$

For all graphs G:

$$\mathbf{X}_G = \sum_{A \subseteq E(G)} (-1)^{\#A} p_{\lambda(A)}.$$

For trees T:

$$\mathbf{X}_T = \sum_{\lambda \vdash n} c_\lambda(T) p_\lambda$$

where

$$c_\lambda = c_\lambda(T) = (-1)^{n-\ell(\lambda)} \#\{A \subseteq E(T) \ \mid \ \lambda(A) = \lambda\}.$$

Elementary Graph Invariants from $\mathbf{X}_G$

- $n = |V(G)| =$ degree of $\mathbf{X}_G$

- $|E(G)| = c_2 = c_{211\cdots 1}$

- $\#$ connected components $=$ min$\{\ell(\lambda) \ \mid \ c_\lambda \neq 0\}$

- $\#$ leaf edges $= c_{n-1}$

- If G is a tree and $k > 1$, then
  number of subtrees of G with k vertices $= c_k$.

$\ldots$

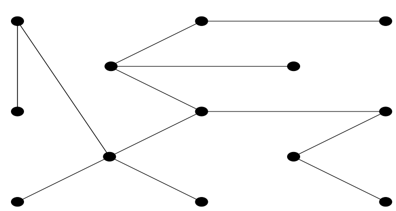
The Subtree and Connector Polynomials

For trees $\emptyset \neq S \subseteq T$, let $L(S) = \{\text{leaf edges of } S\}$.

Subtree polynomial of T:

$$\mathbf{S}_T = \mathbf{S}_T(q,r) = \sum_{\emptyset \neq S \subseteq T} q^{\#S} r^{\#L(S)}$$

For $\emptyset \neq A \subseteq T$, let $K(A)$ be the unique minimal subset of $E(T) - A$ such that $A \cup K(A)$ is a tree.



Connector polynomial of T:

$$\mathbf{K}_T = \mathbf{K}_T(x,y) = \sum_{\emptyset \neq A \subseteq T} x^{\#A} y^{\#K(A)}.$$

Proposition (Chaudhary-Gordon, 1991) The subtree and connector polynomials can be recovered from each other.

Theorem (JLM-Morin-JDW, 2006)

The subtree and connector polynomials of a tree can be recovered from its chromatic symmetric function.

Specifically, let

$$\psi(\lambda, a, b) = (-1)^{a+b} \binom{\ell - 1}{\ell - n + a + b} \sum_{k=1}^{\ell} \binom{\lambda_k - 1}{a}$$

Then

$$\mathbf{K}_T(x, y) \;=\; \sum_{a>0} \sum_{b \geq 0} x^a y^b \sum_{\lambda \vdash n} \psi(\lambda, a, b)\, c_\lambda(T).$$

Equivalently, define a symmetric function Ψ_n by

$$\Psi_n(x, y) = \sum_{a>0} \sum_{b \geq 0} x^a y^b \sum_{\lambda \vdash n} \psi(\lambda, a, b) \frac{p_\lambda}{z_\lambda}.$$

Then

$$\mathbf{K}_T(x, y) = \left\langle \Psi_n(x, y),\ \mathbf{X}_T \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual Hall scalar product on the space of symmetric functions.

Sketch of the Proof

The coefficient of $x^a y^b$ in $\mathbf{K}_T(x,y)$ is

$$\#\{A \subseteq T \;\mid\; \#A = a,\; \#K(A) = b\}$$

which (via manipulatorics) equals

$$\sum_{\lambda \vdash n} (-1)^{a+b+n-\ell(\lambda)} \binom{\ell(\lambda)-1}{\ell(\lambda)-n+a+b} \sum_{\substack{F \subseteq T \\ \lambda(F)=\lambda}} \alpha(F). \qquad (*)$$

where

$$\alpha(F) = \#\{A \;\mid\; \#A = a,\; A \cup K(A) \subseteq F\}.$$

The key observation is that

$$\alpha(F) = \sum_{k=1}^{\ell(\lambda)} \binom{\lambda_k - 1}{a} \qquad (**)$$

This depends only on $\lambda(F)$, so $(*)$ can be rewritten as a linear combination of the $c_\lambda(T)$.

A Positivity Property of Ψ_n

Rewrite Ψ_n in the basis of *homogeneous* symmetric functions h_μ as

$$\Psi_n(x,y) = \sum_{i,j} \sum_{\mu \vdash n} \xi(\mu,i,j) x^i y^j h_\mu$$

where $\xi(\mu,i,j) \in \mathbb{Q}$.

Conjecture: Let $\varepsilon(\mu)$ be the number of parts of μ of even length. Then

$$(-1)^{\varepsilon(\mu)} \xi(\mu,i,j) \geq 0$$

for all partitions μ and integers i,j.

- Easy to verify for small n (using, e.g., Stembridge's `SF` package for Maple).
- In general $\xi(\mu,i,j) \notin \mathbb{Z}$, but it appears that
$$z_\mu \cdot \xi(\mu,i,j) \in \mathbb{Z}.$$

Consequences of the Main Theorem

1. The path and degree sequences of T, i.e., the numbers

$$\pi_i = \#\{\text{paths in } T \text{ with } i \text{ edges}\}$$

and

$$\delta_j = \#\{\text{vertices of } T \text{ of degree } j\}$$

can be recovered from its chromatic symmetric function.

2. Membership in certain families of trees (spiders, caterpillars, ...) can be deduced from $\mathbf{X}_T$.

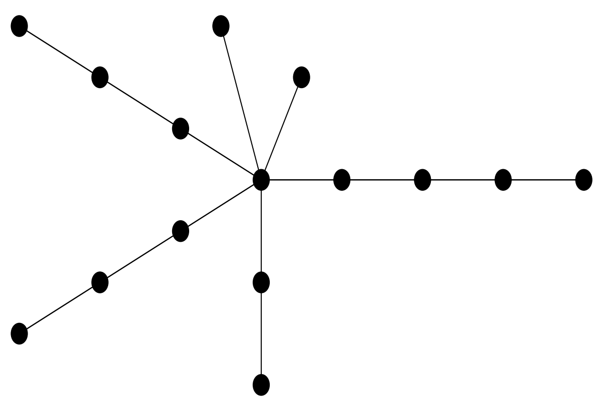
The subtree and connector polynomials do *not* suffice to distinguish trees with $n \geq 11$ (Eisenstat-Gordon, 2006).

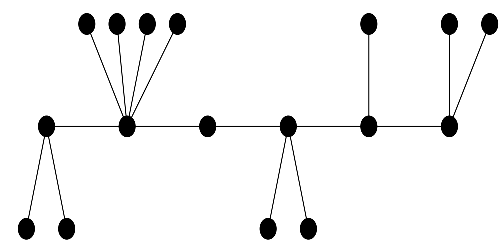
So we still do not know whether the chromatic symmetric function is a complete invariant.

A Little Entomology

A *spider* is a tree with exactly one vertex of degree ≥ 3 (the *torso*).

A *caterpillar* is a tree whose nonleaf vertices form a path (the *spine*).





Theorem (JLM-JDW)

Every spider can be reconstructed from its chromatic symmetric function.

(In fact, from its subtree polynomial; most of the path numbers are elementary symmetric functions of the leg sizes.)

Caterpillars are *not* distinguished by their subtree polynomials; in fact there exist infinitely many counterexamples (Eisenstat-Gordon, 2006), starting at $n = 11$.

Theorem (Morin)
If T is a *symmetric* caterpillar (i.e., it has an automorphism reversing the spine) then it is distinguished by $\mathbf{X}_T$.

Theorem (JLM-JDW-Morin)
If T is a caterpillar in which every spine vertex has a different positive number of adjacent leaves, then it is distinguished by $\mathbf{X}_T$.

Further Questions

- Prove the skew-positivity of $\Psi_n(x, y)$, preferably by finding a combinatorial interpretation for $z_\mu \xi_\mu$.

- Are there other special classes of trees distinguished by their chromatic symmetric function (e.g., binary trees)?

- Does the Eisenstat-Gordon construction of nonisomorphic trees with the same subtree polynomial produce two trees with the same chromatic symmetric function?