The Incidence Hopf Algebra of Graphs

Brandon Humpert <bhumpert@math.ku.edu>
Jeremy L. Martin <jmartin@math.ku.edu>
University of Kansas

FPSAC 2011
Reykjavik, Iceland

Full article: arXiv:1012.4786

Hopf Algebras

A (graded, connected) **Hopf algebra** $\mathcal{H}$ is a graded $\mathbb{C}$-algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$, with $\mathcal{H}_0 = \mathbb{C}$, and maps

$$\begin{aligned} \epsilon &: \mathcal{H} \to \mathbb{C} && (\textit{counit}), \\ \Delta &: \mathcal{H}_n \to \bigoplus_k \mathcal{H}_k \otimes \mathcal{H}_{n-k} && (\textit{comultiplication}) \end{aligned}$$

satisfying various algebraic properties (e.g., coassociativity).

Idea

Comultiplication records decompositions of a combinatorial object into two pieces.

The Graph Hopf Algebra

The **graph Hopf algebra** is $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n$, where $\mathcal{G}_n = \mathbb{C}$-span of isomorphism classes $[G]$ of simple graphs on n vertices, with multiplication $[G][H] = [G \dot\cup H]$ and comultiplication

$$\Delta(G) = \sum_{X \subseteq V(G)} G|_X \otimes G|_{\overline{X}}$$

$$\Delta^{k-1}(G) = \sum_{V(G) = X_1 \dot\cup \cdots \dot\cup X_k} G|_{X_1} \otimes \cdots \otimes G|_{X_k}$$

Properties of the Graph Hopf Algebra

- The multiplicative unit in $\mathcal{G}$ is K_0 (the graph with no vertices).
- The counit is

$$\epsilon(G) = \begin{cases} 1 & \text{if } G = K_0, \\ 0 & \text{if } G \neq K_0. \end{cases}$$

- $\mathcal{G}$ is *cocommutative* — $\Delta(G)$ is unchanged by flipping all tensors
- $\mathcal{G}$ is an *incidence Hopf algebra* [IHA] in the sense of Schmitt [1994] (prototype: Rota's Hopf algebra of graded posets)

Characters

A **character** on $\mathcal{G}$ is a $\mathbb{C}$-linear function $\phi : \mathcal{G} \to \mathbb{C}$ that is multiplicative on connected components and has $\phi(K_0) = 1$.

Definition (Convolution Product of Characters)

$$(\phi * \psi)(h) = \sum \phi(h_1)\psi(h_2)$$

where $\Delta(h) = \sum h_1 \otimes h_2$.

- Characters form a group under convolution.
- The counit ϵ is the identity: $\epsilon * \phi = \phi = \phi * \epsilon$.

Graph Invariants from Characters

Fact

For every character ϕ and element $h \in \mathcal{H}$, the function

$$k \in \mathbb{Z} \;\mapsto\; P_{\phi,h}(k) = \phi^k(h) = \underbrace{(\phi * \cdots * \phi)}_{k \text{ times}}(h)$$

is a polynomial in k.

Idea

Use the Hopf algebra structure of $\mathcal{G}$ to study polynomial invariants of graphs that arise from characters in this way.

Example: The Chromatic Polynomial

Define the character ζ on $\mathcal{G}$ by

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} P_{\zeta,G}(k) \ = \ \zeta^k(G) \ &= \sum_{V(G)=X_1 \dot\cup \cdots \dot\cup X_k} \zeta(G|_{X_1}) \cdots \zeta(G|_{X_k}) \\ &= \text{number of proper } k\text{-colorings of } G \\ &= \text{chromatic polynomial of } G \end{aligned}$$

The Antipode

Every graded connected Hopf algebra has a unique **antipode**: an automorphism $S : \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{aligned} S(h) &= h && \text{for } h \in \mathcal{H}_0, \\ (m \circ (S \otimes Id) \circ \Delta)(h) &= 0 && \text{for } h \in \mathcal{H}_n,\ n > 0. \end{aligned}$$

(These formulas allow S to be calculated recursively, like the Möbius function of a poset.)

Fact

The convolution inverse of a character ϕ is $\phi^{-1} = \phi \circ S$.

A Classic Antipode Formula

Theorem (Schmitt 1994)

$$S(G) = \sum_{\pi \in \mathcal{P}(G)} (-1)^{|\pi|} \, |\pi|! \; G_\pi$$

where: $\mathcal{P}(G)$ = *ordered partitions of* $V(G)$ *into nonempty blocks*
G_π = *disjoint union of induced subgraphs on blocks of* π

- Follows from Schmitt's general antipode formula for any IHA
- Not cancellation-free — different π's can have the same G_π
- Takeuchi (1971) had given an antipode formula for connected (not necessarily graded) Hopf algebras

A New Antipode Formula

Theorem (Humpert–Martin 2010)

$$S(G) = \sum_{F \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F)}\, a(G/F)\, G_{V,F}$$

where: $\mathcal{F}(G)$ = *flats of graphic matroid* M_G *of* G
rk = *rank* = *largest acyclic subset*
a = *number of acyclic orientations*

- Bad news: Specific to $\mathcal{G}$ (does not generalize to other IHAs)
- Good news: Cancellation-free — handy for calculations
- Aguiar–Ardila (unpublished): more general version in the context of Hopf monoids

The Tutte and Rank-Nullity Polynomials

The **Tutte polynomial** and **rank-nullity polynomial** of a graph G are defined by

$$T_G(x,y) = \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}(G)-\mathrm{rk}(A)} (y-1)^{|A|-\mathrm{rk}(A)}$$

$$R_G(x,y) = \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}(A)} (y-1)^{|A|-\mathrm{rk}(A)}$$

- Every graph invariant satisfying a deletion/contraction recurrence (spanning trees, acyclic orientations, chromatic polynomial, ...) is an evaluation of T_G (essentially).

Convolution Powers of the Rank Character

The Tutte and rank-nullity polynomials give characters on $\mathcal{G}$:

$$\tau_{x,y}(G) = T_G(x,y), \qquad \rho_{x,y}(G) = R_G(x,y).$$

Theorem (Humpert–Martin 2010)

$$\rho^k_{x,y}(G) = P_{x,y}(G;k) = k^{n-\mathrm{rk}(G)}(x-1)^{\mathrm{rk}(G)}\,T_G\left(\frac{k+x-1}{x-1}, y\right).$$

Applications

Specializing x, y and k yields formulas like

$$\begin{aligned} (\widetilde{\tau_{0,y}})^{-1} &= \overline{\tau_{2,y}} \\ (\tau_{2,y})^{k}(G) &= k^{c(G)}T_G(k+1,y) \\ (\widetilde{\tau_{0,y}})^{k}(G) &= k^{c(G)}(-1)^{\mathrm{rk}(G)}T_G(1-k,y) \end{aligned}$$

where $\overline{\phi} = (-1)^n\phi$ and $\tilde{\phi} = (-1)^{\mathrm{rk}}\phi$.

Other consequences include

- the expression for the chromatic polynomial in terms of T_G
- Stanley's formula $a(G) = |\chi_G(-1)|$

A Combinatorial Interpretation of $T_G(3,2)$

Corollary

If G is connected, then

$$T(G;3,2) = \frac{(\tau_{2,2} * \tau_{2,2})(G)}{2} = \sum_{X \subseteq V(G)} 2^{e(G|_X)+e(G|_{\overline{X}})-1}$$

$$= \#\left\{ \begin{array}{c} \textit{pairs } (f,A)\textit{, where } f \textit{ is a 2-coloring of } G \\ \textit{and } A \textit{ is a set of monochromatic edges} \end{array} \right\}.$$

(Proof: Set $x = y = k = 2$.)

A Curious (?) Reciprocity Relation

Theorem (Humpert–Martin 2010)

Define the character $\mathbf{1}$ *by* $\mathbf{1}(G) = 1$ *for all* G*. Then*

$$(\mathbf{1} * \zeta^n)(K_m) = (\mathbf{1} * \zeta^m)(K_n).$$

- Idea: $\mathbf{1} * \zeta^n$ counts "near-colorings" of G, in which one color class need not be a coclique. The expression for $(\mathbf{1} * \zeta^n)(K_m)$ is symmetric in m and n.
- Conjecture:
  $(\mathbf{1} * \zeta^{-1})(K_n) = (-1)^n \times$ number of derangements of $[n]$.
- Enumeration of "generalized derangements" via characters?