Pseudodeterminants and perfect square spanning tree counts

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Cellular Trees

X pure cell complex (= CW complex) of dimension d

∂_k cellular boundary map $\partial_k : C_k(X) \to C_{k-1}(X)$

Tree in X: $T = T_d \cup \mathrm{Skel}_{d-1}(X)$ where T_d = column basis of ∂_d

- $H_d(T;\mathbb{Q}) = 0$
- $H_{d-1}(T;\mathbb{Q}) = H_{d-1}(X;\mathbb{Q})$

$\mathscr{T}_{\mathbf{k}}(\mathbf{X})$ = set of all k-trees in X = trees in $\mathrm{Skel}_k(X)$

Examples:

- $\mathscr{T}_1(X) = \{$spanning forests of 1-skeleton graph$\}$
- $\mathscr{T}_0(X) = \{$individual vertices$\}$
- $\mathscr{T}_d(X \cong \mathbb{S}^d) = \{X - \sigma : \ \sigma \text{ a facet}\}$

Counting Cellular Trees

Assume $\tilde{H}_{k-1}(X;\mathbb{Q}) = 0$ (analogue of connectedness).

Tree count:

$$\tau_k(X) = \sum_{T \in \mathscr{T}_k(X)} |\tilde{H}_{k-1}(T;\mathbb{Z})|^2.$$

Weighted tree count: Assign each $\sigma \in X$ a monomial weight $\mathbf{q}_\sigma$.

$$\tau_k(X;\mathbf{q}) = \sum_{T \in \mathscr{T}_k(X)} |\tilde{H}_{k-1}(T;\mathbb{Z})|^2 \prod_{\sigma \in T} \mathbf{q}_\sigma$$

Counting Cellular Trees

Cellular matrix-tree theorem: expresses $\tau_k(X), \tau_k(X; \mathbf{q})$ in terms of eigenvalues/cokernels of combinatorial Laplacians $\partial_k \partial_k^{tr}$.

- Bolker '78: first studied simplicial spanning trees
- Kalai '83: homology-squared weighting; skeletons of simplices
- Adin '92: complete colorful complexes
- Duval–Klivans–JLM; Lyons; Catanzaro–Chernyak–Klein: general formulations

The cellular matrix-tree theorem can be restated in terms of **pseudodeterminants.**

Pseudodeterminants

The cellular matrix-tree theorem can be restated in terms of pseudodeterminants. What's a pseudodeterminant?

Let $L \in \mathbb{Z}^{n \times n}$, not necessarily of full rank; eigenvalues $\lambda_1, \ldots, \lambda_n$.

Pseudodeterminant pdet(L): last nonzero coefficient of characteristic polynomial = coefficient of $t^{n - \operatorname{rank} L}$.

$$\operatorname{pdet} L = \prod_{\lambda_i \neq 0} \lambda_i = \sum_{I \subseteq [n]:\ |I| = \operatorname{rank} L} \det L_{I,I}$$

(So $\operatorname{pdet} L = \det L$ if L is of full rank.)

Counting Trees with Pseudodeterminants

Cellular Matrix-Tree Theorem, Pseudodeterminant Version:
Let $L_k^{ud} = \partial_k \partial_k^{tr}$, the $(k-1)^{th}$ *updown Laplacian* of X. (This is a linear operator on $C_{k-1}(X)$.) Then

$$\operatorname{pdet} L_k^{ud} = \tau_k(X)\tau_{k-1}(X).$$

Classical matrix-tree theorem: G graph, $L = L_0^{ud}(G)$.

$$\#\text{ spanning trees} = \frac{\text{product of nonzero eigenvalues of } L}{\text{number of vertices}}$$

$$\tau_1(G) = \operatorname{pdet} L \,/\, \tau_0(G)$$

Pseudodeterminants and (Skew)-Symmetry

Proposition

Let $\partial \in \mathbb{Z}^{n\times n}$ be either symmetric or skew-symmetric. Then:

1. $\operatorname{pdet}(\partial\partial^{tr}) = (\operatorname{pdet}\partial)^2$.
2. *All principal minors $\partial_{I,I}$ have the same sign, so*

$$\operatorname{pdet}\partial \;=\; \pm\sum_{I} |\operatorname{coker}\partial_{I,I}| \qquad (\bigstar)$$

where I ranges over all row bases of ∂.

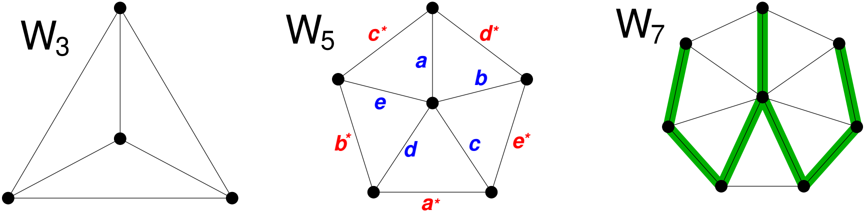
Question

What topological setup will give $(\bigstar)$ combinatorial meaning?

Perfect Square Phenomena in Spanning Tree Counts

Tutte: G planar; $G \cong G^*$ from antipodal map on $\mathbb{S}^2$ $\Longrightarrow$

$$\tau(G) = (\text{number of self-dual spanning trees})^2.$$



$\tau = 16 = 4^2$ $\quad$ $\tau = 121 = 11^2$ $\quad$ $\tau = 841 = 29^2$

Question

Are there analogous perfect-square phenomena for higher-dimensional self-dual cell complexes?

Even-Dimensional Spheres: Maxwell's Theorem

Theorem (Maxwell '09)

Let k be odd. Let X be an antipodally self-dual cellular $\mathbb{S}^{2k}$ with at least one $\mathbb{Z}$-acyclic self-dual tree. Then

$$\underbrace{\sum_{T \in \mathscr{T}_k(X)} |\tilde{H}_{k-1}(T;\mathbb{Z})|^2}_{\tau_k(X)} = \left(\sum_{\substack{T \in \mathscr{T}_k(X) \\ T \text{ self-dual}}} |\tilde{H}_{k-1}(T;\mathbb{Z})| \right)^2 .$$

What about **odd-dimensional** antipodally self-dual spheres?

- $\dim = 2k$: involution on k-dimensional faces
- $\dim = 2k+1$: pairing between k- and $(k+1)$-dim'l faces

Self-Dual Cell Complexes

Self-dual d-ball: regular cell complex $B \cong \mathbb{B}^d$, with an anti-automorphism α of its face poset:

$$\sigma \subseteq \tau \iff \alpha(\sigma) \supseteq \alpha(\tau).$$

Self-dual (d − 1)-sphere: $S = \partial B \cong \mathbb{S}^{d-1}$.

Example: $B =$ simplex on vertex set V; $\quad \alpha(\sigma) = V \setminus \sigma$

Example: Self-dual polytopes (polygons in $\mathbb{R}^2$; pyramids over polygons in $\mathbb{R}^3$; the 24-cell in $\mathbb{R}^4$; ...)

Tree Counts in Self-Dual Complexes

Proposition

Let B be a self-dual cellular $\mathbb{B}^d$ and $j + k = d - 1$. Then $\tau_j(B) = \tau_k(B)$.

Proof sketch.

For $T \in \mathscr{T}_j(B)$, consider the Alexander dual

$$T^\vee = \{\sigma \in B \colon \alpha(\sigma) \notin T\}.$$

Then

$$\mathscr{T}_j(B) = \{\, T^\vee \colon\ T \in \mathscr{T}_k(B)\}$$

and

$$H_{j-1}(T;\mathbb{Z}) \cong H_{k-1}(T^\vee;\mathbb{Z}).$$

□

Perfect Square Phenomenon for Even-Dimensional Balls

Let $B \cong \mathbb{B}^{2k}$ be self-dual and let $\partial = \partial_k(B)$. Then

$$\tau_{k-1}(B) = \tau_k(B)$$

and the pseudodeterminant version of the CMTT says that

$$\operatorname{pdet}(\partial\partial^{tr}) = \tau_{k-1}(X)\tau_k(X) = \tau_k(X)^2.$$

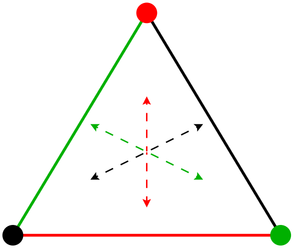
Repeat Question: What additional structure will enable

$$\operatorname{pdet}\partial \;=\; \pm\sum_{I} |\operatorname{coker}\partial_{I,I}| \qquad (\bigstar)$$

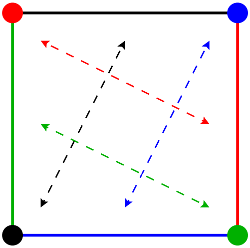
to carry combinatorial meaning?

Antipodally Self-Dual Complexes

Definition: A self-dual cellular d-ball (B, α) is **antipodally self-dual** if α arises from the antipodal map on $\partial B \cong \mathbb{S}^{d-1}$.



Antipodal self–duality



Non–antipodal self–duality

Technical details: explicit orientations, dual block complex, Poincaré duality. . .

Antipodal Self-Duality and Orientations

Proposition (Very Technical!)

Let B be an antipodally self-dual cellular $(2k)$-ball. Then B can be oriented so that the middle boundary matrix ∂_k satisfies

$$\partial^{tr} = (-1)^k \partial.$$

Example

If B is the simplex on vertices $[2k+1]$, then start with the "textbook" orientation and reorient:

$$\sigma = \{v_0, \dots, v_k\} \in B_k \ \mapsto \ (-1)^{\sum v_i} \sigma.$$

Antipodally Self-Dual Even-Dimensional Balls

Proposition

Let $B \cong \mathbb{B}^{2k}$ be antipodally self-dual. Then B can be oriented so that the middle boundary matrix $\partial = \partial_k$ satisfies

$$\partial^{tr} = (-1)^k \partial.$$

Theorem

Let $B \cong \mathbb{B}^{2k}$ be antipodally self-dual and write $\tau_i = \tau_i(B)$. Then

$$\tau_k = \tau_{k-1} \mathbin{\color{red}{\mathbf{=}}} \operatorname{pdet} \partial \underset{\star}{=} \sum_I |\operatorname{coker} \partial_{I,I}| = \sum_{T \in \mathscr{T}_k(S)} |H_k(T, T^\vee; \mathbb{Z})|.$$

(There is also a **q**-analogue.)

Open Questions

1. What about antipodally self-dual $\mathbb{B}^d$ with $d \equiv 1 \pmod 4$?
   - $d \equiv 3 \pmod 4$: Maxwell
   - $d \equiv 0, 2 \pmod 4$: this work
2. Any hope of bijective proofs?
   - E.g., higher-dimensional Prüfer code, Joyal bijection, ...

Thanks for listening!

Appendix A: The Weighted CMTTPV

Weighted Cellular Matrix-Tree Theorem, Pdet Version

Ingredients:

S	cell complex of dimension $\geq k$	$\partial = \partial_k$
$\mathbf{x} = (x_i)$	variables indexing $(k-1)$-cells	$X = \mathrm{diag}(\mathbf{x})$
$\mathbf{y} = (y_i)$	variables indexing k-cells	$Y = \mathrm{diag}(\mathbf{y})$

Formula:

$$\mathrm{pdet}(X^{1/2} \cdot \partial \cdot Z \cdot \partial^{tr} \cdot Y^{1/2}) \; = \; \tau_k(S; \mathbf{y}) \, \tau_{k-1}(S; \mathbf{z}^{-1}).$$

Setting $y_i = z_i = 1$ recovers the unweighted formula.

Appendix B: A Little Linear Algebra

∂: matrix of rank r
I, I': sets of r rows
J, J': sets of r columns

Useful Fact 1 ("The Minor Miracle")
I and J are a row basis and a column basis respectively if **and only if** $\det \partial_{I,J} \neq 0$.

Useful Fact 2

$$\det \partial_{I,J} \det \partial_{I',J'} = \det \partial_{I,J'} \det \partial_{I',J}.$$

Important consequences for matrices that are (skew-)symmetric!

Appendix C: Explicit Reorientation of Simplices

	123	**124**	125	134	**135**	145	**234**	235	**245**	345
45	0	0	0	0	0	−	0	0	−	−
35	0	0	0	0	−	0	0	−	0	+
34	0	0	0	−	0	0	−	0	0	−
25	0	0	−	0	0	0	0	+	+	0
24	0	−	0	0	0	0	−	0	−	0
23	−	0	0	0	0	0	−	−	0	0
15	0	0	+	0	+	+	0	0	0	0
14	0	+	0	+	0	−	0	0	0	0
13	+	0	0	−	−	0	0	0	0	0
12	−	−	−	0	0	0	0	0	0	0

Appendix C: Explicit Reorientation of Simplices

	123	**124**	125	134	**135**	145	**234**	235	**245**	345
45	0	0	0	0	0	−	0	0	+	−
35	0	0	0	0	+	0	0	−	0	+
34	0	0	0	−	0	0	+	0	0	−
25	0	0	−	0	0	0	0	+	−	0
24	0	+	0	0	0	0	+	0	+	0
23	−	0	0	0	0	0	+	−	0	0
15	0	0	+	0	−	+	0	0	0	0
14	0	−	0	+	0	−	0	0	0	0
13	+	0	0	−	+	0	0	0	0	0
12	−	+	−	0	0	0	0	0	0	0