Arithmetical structures on graphs and Catalan combinatorics

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Background: Graph Laplacians and Critical Groups

Let G be a connected graph on vertex set $[n]$ with no loops.
The adjacency matrix $A = A(G)$ is given by

$$a_{ij} = \#\{\text{edges from } i \text{ to } j\}, \qquad i,j \in [n].$$

The Laplacian matrix $L = L(G)$ is given by

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{for } i = j, \\ -a_{ij} & \text{for } i \neq j, \end{cases} \qquad i,j \in [n].$$

That is, $L = D - A$, where $D =$ diagonal matrix of vertex degrees.

Background: Graph Laplacians and Critical Groups

Some standard facts about the Laplacian:

- $\operatorname{rank} L = n - 1$
- $\ker L$ is one-dimensional, spanned by the all-ones vector $\mathbf{1}$.
- $\mathbb{Z}^n / \operatorname{im} L \cong \mathbb{Z} \oplus K(G)$, where $K(G)$, the critical group, has cardinality equal to the number of spanning trees of G.

Idea: Replace L by another singular matrix of the form $D' - A$, where D' is a diagonal matrix.

Arithmetic Structures

Definition (Lorenzini, 1989)

An arithmetical graph consists of a connected graph G on $[n]$ and two vectors $\mathbf{d}, \mathbf{r} \in \mathbb{N}^n_{>0}$ with $\gcd(r_i) = 1$ such that

$$\underbrace{(\operatorname{diag}(\mathbf{d}) - A(G))}_{\tilde{L}} \mathbf{r} = 0.$$

- If $\mathbf{d} = \mathbf{deg}(G)$ and $\mathbf{r} = \mathbf{1}$ then $\tilde{L}$ is the usual Laplacian.

Definition

Let $(G, \mathbf{d}, \mathbf{r})$ be an arithmetical graph.
The critical group $K(G, \mathbf{d}, \mathbf{r})$ is the torsion summand of coker $\tilde{L}$.

Arithmetic Structures

- Motivation from algebraic geometry (Lorenzini '89): study curves C that degenerate into n components $C_1, \ldots, C_n$ with $|C_i \cap C_j| = a_{ij}$.
  - Entries of $\mathbf{d}$'s are self-intersection numbers
  - Critical group $K(G, \mathbf{d}, \mathbf{r})$ = group of components of the Néron model of the Jacobian of the curve
- Lorenzini: "...by presenting here our results without any reference to geometry, some non algebraic geometers will take interest in this subject and bring new techniques to the study of these matrices."

Arithmetic Structures: The Basics

Basic facts about arithmetic graphs (observed by Lorenzini):

Fact 1: *Each of* $\mathbf{d}$ *or* $\mathbf{r}$ *determines the other.*

- Either of $\mathbf{d}, \mathbf{r}$ defines an arithmetic structure on G.
- The set of all arithmetic structures on G is written $\mathrm{Arith}(G)$.

Fact 2: *The "pseudo-Laplacian"* $\tilde{L} = D - A$ *has rank* $n - 1$*, and is an M-matrix in the sense of numerical analysis.*

- Every principal minor of M has positive determinant
- Chip-firing on M-matrices: Guzmán and Klivans, 2015

Arithmetic Structures: The Basics

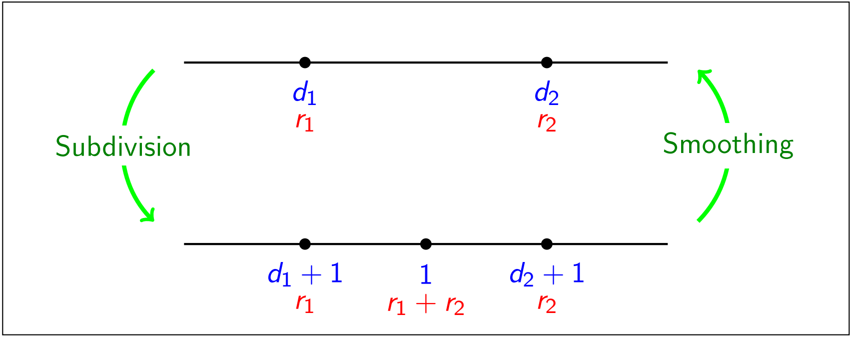
Fact 3: *Every graph has at most finitely many arithmetical structures.*

Lorenzini's proof was general and non-constructive (essentially by reduction to Dickson's lemma).

How many are there?

Subdivision and Smoothing

A degree-2 vertex of an arithmetical graph can be added or deleted:



These operations are key to studying arithmetical structures on paths and cycles (where all vertices have degree ≤ 2).

Example: Arithmetic Structures on the Path $\mathcal{P}_4$

Let $\mathcal{P}_4$ be the path with four vertices. 

An arithmetic structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{P}_4$ is defined by

$$\begin{bmatrix} d_1 & -1 & 0 & 0 \\ -1 & d_2 & -1 & 0 \\ 0 & -1 & d_3 & -1 \\ 0 & 0 & -1 & d_4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = 0 \qquad \text{i.e.,} \qquad \begin{array}{rcl} d_1 r_1 & = & r_2, \\ d_2 r_2 & = & r_1 + r_3, \\ d_3 r_3 & = & r_2 + r_4, \\ d_4 r_4 & = & r_3. \end{array}$$

- $\gcd(\mathbf{r}) = 1$ plus first and last equations $\implies$ $r_1 = r_4 = 1.$
- The two middle equations are equivalent to

$$r_2 \mid r_1 + r_3, \quad r_3 \mid r_2 + r_4.$$

Arithmetic Structures on the Path $\mathcal{P}_n$

$n = 2$

d	**r**
11	11

$n = 3$

d	**r**
121	111
212	121

$n = 4$

d	**r**
1221	1111
2131	1211
1312	1121
2213	1231
3122	1321

$n = 5$

d	**r**
12221	11111
21321	12111
13131	11211
12312	11121
21412	12121
31231	13211
22141	12311
13213	11231
14122	11321
41222	14321
22214	12341
32132	13521
23123	12531
31313	13231

Arithmetic Structures on $\mathcal{P}_n$

Proposition (Oaxaca Group 2016+)

A sequence $(r_1, \ldots, r_n)$ *is an arithmetic r-structure on* $\mathcal{P}_n$ *if and only if* $r_1 = 1$, $r_n = 1$, *and* $r_i | r_{i-1} + r_{i+1}$ *for* $2 \leq i \leq n-1$. *In particular,*

$$|\text{Arith}(\mathcal{P}_n)| = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

- Interpretation #92 in Stanley's *Catalan Numbers*
- Finer enumeration of Arith($\mathcal{P}_n$) reveals more Catalan combinatorics

Arithmetic Structures on $\mathcal{P}_n$

For $(\mathbf{d}, \mathbf{r}) \in \text{Arith}(\mathcal{P}_n)$, let $\mathbf{r}(1) = \#\{i : \; r_i = 1\}$.

Theorem (Oaxaca Group 2016+)

1. *Every* $(\mathbf{d}, \mathbf{r}) \in \text{Arith}(\mathcal{P}_n)$ *has trivial critical group.*
2. *Every* $(\mathbf{d}, \mathbf{r}) \in \text{Arith}(\mathcal{P}_n)$ *satisfies* $\mathbf{r}(1) = 3n - 2 - \sum_{j=1}^{n} d_j$.
3. *For every* $k \in [n]$*, the number of arithmetic structures* $(\mathbf{d}, \mathbf{r})$ *with* $\mathbf{r}(1) = k$ *is given by the* ***ballot number***

$$B(n-2, n-k) = \frac{k-1}{n-1}\binom{2n-2-k}{n-2}$$

(the number of lattice paths from $(0,0)$ *to* $(n-2, n-k)$ *that do not cross above the line* $y = x$*).*

Arithmetic Structures on $\mathcal{P}_n$

Theorem (OG 2016+)

The entries of $\mathbf{d}$ *are distributed identically. Specifically, for every* $i, k \in [n]$*, the number*

$$\#\left\{(\mathbf{d}, \mathbf{r}) \in \mathrm{Arith}(\mathcal{P}_n) \mid d_i = n - k - 1\right\}$$

is given by the ballot number $B(n-2, k)$.

Arithmetic Structures on $\mathcal{C}_n$

Let $\mathcal{C}_n$ be the cycle on $n \geq 2$ vertices.

Similarly to the path, the arithmetic r-structures on $\mathcal{C}_n$ are characterized by the conditions

$$r_i \mid r_{i-1} + r_{i-1} \qquad \forall i \in [n]$$

(taking indices modulo n).

Subdividing and smoothing are defined similarly.

Arithmetic Structures on $\mathcal{C}_n$

Here are all the arithmetic structures on $\mathcal{C}_2$ for $n = 2, 3, 4$, up to dihedral symmetry:

$n = 2$

d	**r**
22	11

Total: 1

$n = 3$

d	**r**	#
222	111	1
331	112	3
521	123	6

Total: 10

$n = 4$

d	**r**	#
2222	1111	1
3231	1112	4
4141	1212	2
4321	1123	8
6221	1234	8
6131	1323	4
5213	1352	8

Total: 35

$|\operatorname{Arith}(\mathcal{C}_5)| = 126$
$|\operatorname{Arith}(\mathcal{C}_6)| = 462$

Arithmetic Structures on $\mathcal{C}_n$

Theorem (Corrales–Valencia 2016+; Lorenzini)

Let $(\mathbf{d}, \mathbf{r})$ *be an arithmetic d-structure on* $\mathcal{C}_n$*. Then:*

1. *Either* $\mathbf{d} = \mathbf{2}$ *or* $\min(d_i) = 1$*.*
2. *If* $\mathbf{d}$ *has an "isolated 1," i.e.,* $d_{i-1} > d_i = 1 < d_{i+1}$*, then*
   (a) $(\mathbf{d}, \mathbf{r})$ *is the subdivision of some* $(\mathbf{d}', \mathbf{r}') \in \mathrm{Arith}(\mathcal{C}_{n-1})$*.*
   (b) $K(\mathcal{C}_n, \mathbf{d}, \mathbf{r}) \cong K(\mathcal{C}_{n-1}, \mathbf{d}', \mathbf{r}')$*.*

Theorem (OG 2016+)

$\mathbf{r}(1) = 3n - \sum_{i=1}^{n} d_i$*, and* $K(\mathcal{C}_n, \mathbf{d}, \mathbf{r})$ *is cyclic of this order.*

Theorem (OG 2016+)

There is a bijection between arithmetic structures $(\mathbf{d}, \mathbf{r})$ *on* $\mathcal{C}_n$ *with* $\mathbf{r}(1) = k$ *and multisubsets of* $[n]$ *of cardinality* $n - k$.

In particular

$$\#\{(\mathbf{d}, \mathbf{r}) \in \mathrm{Arith}(\mathcal{C}_n) \mid \mathbf{r}(1) = k\} = \binom{2n-k-1}{n-k}$$

and

$$\#\,\mathrm{Arith}(\mathcal{C}_n) = \binom{2n-1}{n-1}.$$

Arithmetic Structures on $\mathcal{C}_n$

Theorem (OG 2016+)

There is a bijection between arithmetic structures $(\mathbf{d}, \mathbf{r})$ *on* $\mathcal{C}_n$ *with* $\mathbf{r}(1) = k$ *and multisubsets of* $[n]$ *of cardinality* $n - k$*.*

Proof #1 ("United Airlines Bijection"): explicit algorithm; equivariant w/r/t actions of $\mathbb{Z}_n$ on $\mathcal{C}_n$ by rotation and on multisets by addition modulo n.

Proof #2: idea is to "snip" a structure on $\mathcal{C}_n$ at one of its 1's to obtain a structure on $\mathcal{P}_n$, then reuse what we know about paths.

Arithmetic Structures on Other Graphs

It is much harder to count arithmetic structures for graphs other than $\mathcal{P}_n$ and $\mathcal{C}_n$.

$\mathcal{D}_n$ (Coxeter graph of type D_n — path with branch at end): We have some computations but not enough for a conjecture.

K_n (complete graph): d-structures are positive integer solutions to $1/d_1 + \cdots + 1/d_n = 1$ ("weak Egyptian fractions")

n	1	2	3	4	5	$\cdots$
$\#\,\mathrm{Arith}(K_n)$	1	1	10	215	12231	$\cdots$

(OEIS #A002967; very little known.)

Thank you!

References

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