Informal Seminar on Stanley-Reisner Theory, UMN, Fall 2002
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Introduction and motivation for Stanley-Reisner rings, III
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Theorem (UBC for polytopes, Motzkin, 1956): Let Δ be the boundary of a simplicial convex d-polytope (i.e., a $(d-1)$-sphere) with $f_0 = n$ vertices. Then

$$f_i(\Delta) \leq f_i(\Delta_{C(n,d)}),$$

where $\Delta_{C(n,d)}$ is the boundary of the cyclic d-polytope with n vertices $C(n,d)$.

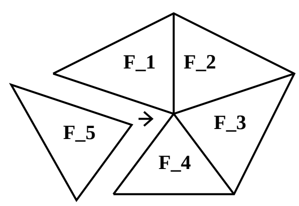
(Note: the combinatorial structure of $C(n,d)$ is completely determined by n and d.)

McMullen's observations (1970):

Observation 1: UBC follows if one can show $H_i(\Delta) \leq \binom{n-d+i-1}{i}$ for all i.

Observation 2: This is easy to show by induction for *shellable* simplicial complexes (and Brugesser and Mani had just shown that boundaries of convex polytopes *are* shellable). Thus UBC is true for convex polytopes.

(Recall that a simplicial complex is *shellable* if it can be built up by adding maximal faces in some order so that at each step, the intersection of the next face with all the previous faces is pure of codimension 1.)



Obeservation 3: $h_i(\Delta)$ = the number of factes in the shelling that have i "old" walls and $d-i$ "new" walls.

Corollary 1: For Δ shellable, $h_i(\Delta) \geq 0$ for all i.

Corollary 2: For Δ the boundary of a *simplicial* convex d-polytope, one has $h_i(\Delta) = h_{d-i}(\Delta)$ for all i.

(Note: These are the Dehn (1905) - Sommerville (1927) Equations. The original proof was topological. $h_0 = h_d$ is just Euler's formula, since h_d is, up to a sign, the Euler characteristic.)

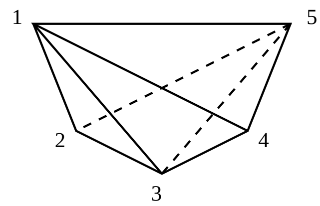
Proof of Corollary 2: The reverse of a Brugesser-Mani shelling is also a shelling, with the notions of "old" and "new" walls reversed.

(Note: An arbitrary convex polytope can always be made simplicial by triangulating the non-simplicial faces. This just adds more faces.)

Question: What about simplicial spheres that are not necessarily polytopal?

We can try to interpret $h_i(\Delta)$ differently, using $k[\Delta]$.

Example:



The maximal non-faces are $x_1x_3x_5$ and x_2x_4, so

$$R = k[x_1, x_2, x_3, x_4, x_5]/(x_1x_3x_5, x_2x_4)$$

(note that R has Krull dimension 3), and

$$Hilb(R,t) = \frac{1+2t+2t^2+t^3}{(1-t)^3}.$$

It turns out that the three linear forms

$$\begin{aligned}\Theta_1 &= x_3 - x_1\\ \Theta_2 &= x_5 - x_1\\ \Theta_3 &= x_4 - x_2\end{aligned}$$

form an R-regular sequence, i.e.,

Θ_1 is a non zero-divisor of R,
Θ_2 is a non zero-divisor of $R/(\Theta_1)$,
Θ_3 is a non zero-divisor of $R/(\Theta_1, \Theta_2)$.

So

$$\begin{aligned} R/(\Theta) &:= R/(\Theta_1, \Theta_2, \Theta_3)\\ &= K[x_1, x_2, x_3, x_4, x_5]/(x_1x_3x_5, x_2x_4, x_3 - x_1, x_5 - x_1, x_4 - x_2)\\ &= k[y_1, y_2]/(y_1^3, y_2^2), \text{ where } y_1 = x_1, y_2 = x_2\\ &= k\text{-span of } \{1, y_1, y_2, y_1^2, y_1y_2, y_1^2y_2\}.\end{aligned}$$

Note that the basis has one element of degree 0, two of degree 1, two of degree 2 and one of degree 3. This matches the h-vector $h(\Delta) = (1, 2, 2, 1)$ we computed earlier. This is not a coincidence!

Fact: If R is a graded ring and Θ a homogeneous non zero-divisor of degree m, then

$$\text{Hilb}(R/(\Theta), t) = (1 - t^m)\text{Hilb}(R, t).$$

(Note: In our example, $m = 1$ and we performed this process three times, to get $(1-t)^3 \frac{1+2t+2t^2+t^3}{(1-t)^3} = 1 + 2t + 2t^2 + t^3$.)

Reason: Multiplication by Θ gives us a short exact sequence

$$0 \to R(-m) \to R \to R/(\Theta) \to 0.$$

($R(-m)$ is a free R-module whose basis element is degree m.)

The corresponding Hilbert series are $t^m\mathrm{Hilb}(R,t)$, $\mathrm{Hilb}(R,t)$, and $\mathrm{Hilb}(R/(\Theta),t)$, and by exactness we get

$$t^m\mathrm{Hilb}(R,t) - \mathrm{Hilb}(R,t) + \mathrm{Hilb}(R/(\Theta),t) = 0.$$

Definition: A finitely generated k-algebra R of Krull dimension d is *Cohen-Macaulay* (CM) if there exists an R-regular sequence $\Theta_1, \Theta_2, \ldots, \Theta_d$.

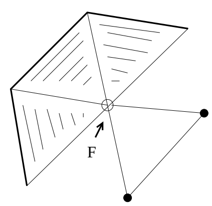
Fact 1: If R is CM and generated by elements of degree 1 (this is the case for a SR ring) then one can always choose $\Theta_1, \Theta_2, \ldots, \Theta_d$ of degree 1, as long as k is "big enough" (which we can ensure by enlarging k to $|k| = \infty$ without changing the characteristic).

Note: This is similar to Noether Normalization, but stronger.

Note: In a CM ring, *every* system of parameters is a regular sequence.

Fact 2 (Reisner's Theorem, Minnesota PhD thesis, 1976): For Δ a d-dimensional complex, $k[\Delta]$ is CM if and only if for every face F of Δ, the link has no reduced homology below dimension $d-1-\dim F$, i.e., if and only if $H_i(\mathrm{link}_\Delta(F);k) = 0$ for $i < d-1-\dim F$. By Munkries, this happens if and only if $H_i(||\Delta||;k) = 0$ for $i < d$, which is if and only if $H_i(||\Delta||, ||\Delta|| - x) = 0$ for $i < d$ and all $x \in ||\Delta||$.

Note: The "star" of a face F is a neighborhood of F. The link is the set of all faces disjoint from F, i.e., the base of the star.



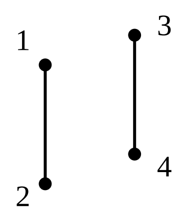
Corollary 1: Δ a simplicial sphere implies $k[\Delta]$ is CM for all fields k

Corollary 2 (UBC for spheres, Stanley, 1975): Δ a CM simplicial complex implies

$$0 \le h_i(\Delta) \le \binom{n-d+i-1}{i}.$$

Proof of Corollary 2: $h_i(\Delta) = dim_k(R/(\Theta))_i$, so the quotient ring is isomorphic to a polynomial ring in $n-d$ variables quotient by some ideal. Hence $h_i(\Delta) \le \dim_k(k[y_1, y_2 \ldots, y_{n-d}]_i) = \binom{n-d+i-1}{i}$ (the number of monomials of degree i from $n-d$ variables).

Example of a non-CM complex:



$$f = (f_{-1}, f_0, f_1) = (1, 4, 2)$$
$$h = (h_0, h_1, h_2) = (1, 2, -1)$$

Set $R = k[x_1, x_2, x_3, x_4]/(x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. We can find one non zero-divisor, say $x_1 - x_3$, and quotient by $x_1 - x_3$ to get $k[x_1, x_2, x_4]/(x_1^2, x_1x_4, x_1x_2, x_2x_4)$. This ring has no non zero-divisors since, for example, everything of positive degree kills x_1.

Note: In this example R is the coordinate ring of two 2-dimensional hyperplanes in 4-space, meeting at a single point. It is a standard example of a non-CM variety with equidimensionality. See Eisenbud's book.