Stanley-Reisner Rings (10/24/02)

Speaker: Vic Reiner

$k(\Delta)$ associated a simplicial complex Δ on vertex set $V = k[x_v \ : \ v \in V]/I_\Delta$, where

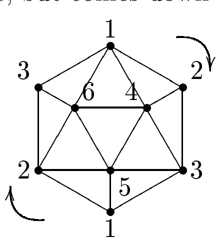
$$I_\Delta = \{x_{v_1}, \ldots, x_{v_r} \ : \ \{v_1, \ldots, v_r\} \notin \Delta\}$$
$$= \text{arbitrary square-free monimial ideal}$$

Motivation (i)

Arbitrary graded rings deform to $k[\Delta]$'s, leaving many properties (Knull dimension, Hilbert series, degree of projection embedding) unchanged; and having many homological invariants only increasing.

Motivation (ii)

For $k[d]$, almost any (ring-theoretic) homological invariant (e.g., $Tor^s(k[\Delta],)$, $H_m(k[\Delta])$ local cohomology) are computed via simplicial (co-) homology of Δ. E.g., dependence on the characteristic of the field k can be subtle for these ring invariants, but comes down to torsion for $H(\Delta, k)$.

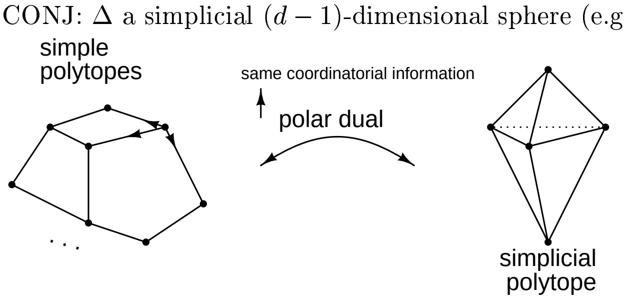
$\Delta =$ 

$= \mathbb{R}P^2$ has $k[\Delta] = k[x_1, x_2, \ldots, x_6]/(x_1x_2x_3, x_1x_2x_6, \ldots)$
with most of its homological invariants
depending upon whether $\text{char}(k) = 2$ or not, since

$$\tilde{H}_i(\Delta; k) = \begin{cases} 0 & i > 2 \\ k & i = 2 \\ k & i = 1 \\ 0 & i = 0 \end{cases} \quad \text{if char}(k) = 2$$
$$= 0 \quad \forall i, \qquad \text{if char}(k) \neq 2.$$

Motivation (iii)

For some combinatorial problems about simplicial complexes Δ, the approach via $k[\Delta]$ is the easy way or the only way. E.g., The upper bound conjecture (UBC) for simplicial polytopes and spheres (Motzkin 1957?)
CONJ: Δ a simplicial $(d-1)$-dimensional sphere (e.g., boundary of a simplicial convex polytope)

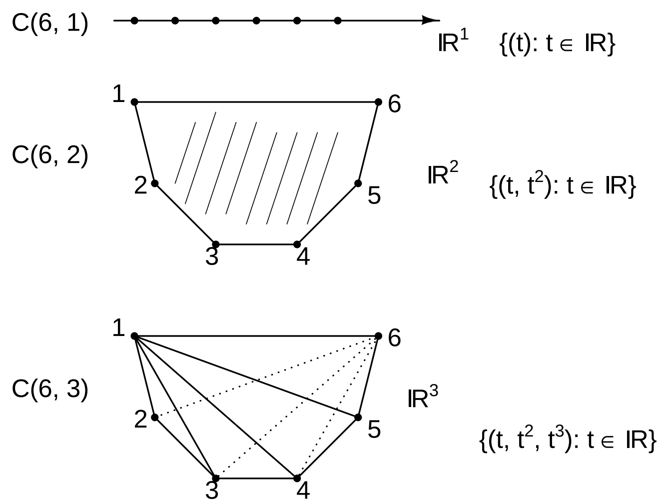


$\Rightarrow \underbrace{f_i(\Delta)}_{\#\text{ i-dim'l simplicies}} \leq f_i(\Delta_{c(n,d)})$

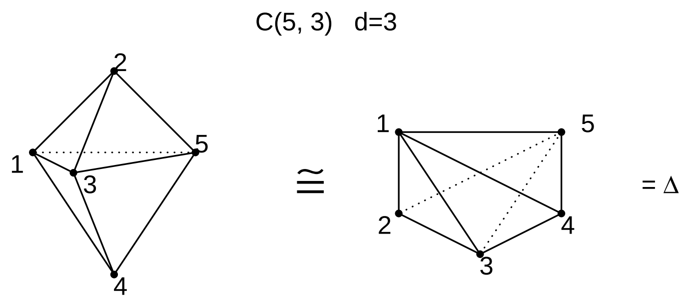
where

$$\Delta_{c(n,d)} = \text{boundary of the cyclic } d\text{-polytope } C(n,d) \text{ with } n \text{ vertices}$$
$$= \text{convex hull of any } n \text{ points on the moment curve } \{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\} \subset \mathbb{R}^d$$

e.g. $n = 6$



UBC is proven for convex polytopes by Peter Mcmullen in 1970 (?) using key observations about the n-vectors ...



$$f(\Delta)(f_{-1}, f_0, f_1, f_2) = (1, 5, 9, 6)$$
$$h(\Delta)(h_0, h_1, h_2, h_3) = (1, 2, 2, 1)$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 5 & & \\ & 1 & & 4 & & 9 & \\ 1 & & 3 & & 5 & & 6 \\ \hline (1 & & 2 & & 2 & & 1\ ) = h(\Delta) \end{array}$$

So

$$\begin{aligned}
\mathrm{Hilb}(k[\Delta],t) &= f_{-1} + f_0\left(\frac{t}{1-t}\right) + f_1\left(\frac{t}{1-t}\right)^2 + f_2\left(\frac{t}{1-t}\right)^3 \\
&= 1 + 5\left(\frac{t}{1-t}\right) + 9\left(\frac{t}{1-t}\right)^2 + 6\left(\frac{t}{1-t}\right)^3 \\
&= \frac{h_0 + h_1 t + h_2 t^2 + f_3 t^3}{(1-t)^3} \\
&= \frac{1 + 2t + 2t^2 + t^3}{(1-t)^3}
\end{aligned}$$

McMullen's observation 1

UBC follows from

$$h_i(\Delta) \le \binom{n-d+i-1}{i}$$

where $n = f_0 = \#$ of vertices.

(follows from explicit knowledge of f_i for boundary of $C(n,d)$ and a little mucking around...)

McMullen's observation 2

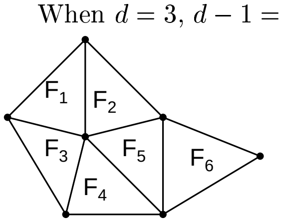
$h_i(\Delta) \le \binom{n-d+i-1}{i}$ is easy to prove by induction on $f_{d-1} = \#$ of *facets* (=maximal faces) for Δ which are pure shellable simplicial complies (of dimension $d-1$ with n vertices)

Δ is *shellable* if it can be built up by ordering facets $F_1, F_2, \ldots$ so that $\forall i \ge 2$,

$$F_i \cap \underbrace{\left(\overline{\cup_{j<i} F_j}\right)}_{\text{sub complex gen'd by } F_1, F_2, \ldots, F_{i-1}}$$

is pure of codimension inside F_i

When $d = 3$, $d - 1 = 2$,

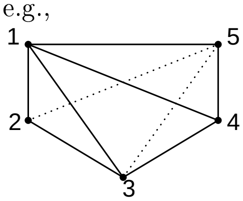


Brngesser & Mani (1969?), Boundary of convex polytopes are shellable (this proves UBC)

McMullen's observation 3

For Δ shellable, $h_i(\Delta)$ counts something: it is equal to the number of facets F_i is shelling having $d - i$ new walls, i old walls, where $d - i$ new walls are not in $\overline{\cup_{j<i} F_i}$.

e.g.,



	facets	new walls	d: # new walls	
F_1	123	12, 13, 23	0	$\}h_0 = 1$
F_2	134	14, 34	1	$\}h_1 = 2$
F_3	145	15, 45	1	
F_4	345	35	2	$\}h_2 = 2$
F_5	235	25	2	
F_6	125	$\emptyset$	3	$\}h_3 = 1$

For shellable Δ,

Cor 1: $h_i(\Delta) \geq 0$

Cor 2: $h_i(\Delta) = h_{d-i}(\Delta)$ (provided Δ is the boundary of a d-dimensional polytope, or more generally has a shelling order whose reverse is also a shelling order).

$\overbrace{\text{Dehn}}^{1905} - \overbrace{\text{Sommerville}}^{1927}$ equations. (The reverse of a Barg-Mani shelling is still a shelling, and "old" $\leftrightarrow$ "new")