Hopf Monoids II

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The Antipode in $\mathbf{L}$

Recall that $\boldsymbol{\ell}$ is the set species defined by

$$\boldsymbol{\ell}[I] = \{\text{linear orders on } I\} = \{\text{bijections } [|I|] \to I\}.$$

The corresponding vector species $\mathbf{L} = \mathbb{k}\boldsymbol{\ell}$ can be made into a Hopf monoid by

$$\mu_{I,J}(u \otimes v) = u \cdot v, \qquad \Delta_{I,J}(w) = w|_I \otimes w|_J$$

where $\cdot$ means concatenation.

Example

$\mu(528 \otimes 74639) = 52874639$
$\Delta_{2345,6789}(52874639) = 5243 \otimes 8769$

The Antipode in $\mathbf{L}$

Let $w \in \mathbf{L}[n]$. Takeuchi's formula gives

$$\mathbf{s}(w) = \sum_{\Phi \vDash [n]} (-1)^{|\Phi|} \mu_\Phi(\Delta_\Phi(w)) = \sum_{\Phi = \Phi_1|\cdots|\Phi_k \vDash [n]} (-1)^k \underbrace{\mu_\Phi(\Delta_\Phi(w))}_{u_\Phi}.$$

Say that a *w-split* of $u \in \mathbf{L}(E)$ is an expression $u = u^{(1)} \cdot \cdots \cdot u^{(k)}$, such that each $u^{(i)}$ is correctly ordered w/r/t w.

Example

$w = 123456789$, $u = 738146295$

$7 \cdot 38 \cdot 14 \cdot 6 \cdot 29 \cdot 5$:	w-split
$7 \cdot 3 \cdot 8 \cdot 146 \cdot 2 \cdot 9 \cdot 5$:	w-split
$7 \cdot 38 \cdot 14 \cdot \underline{62} \cdot 9 \cdot 5$:	not a w-split

The Antipode in $\mathbf{L}$

Let $\Phi \models [n]$. Then $\mu_\Phi(\Delta_\Phi(w)) = w|_{\Phi_1} \cdots w|_{\Phi_k}$ is a w-split. So

$$\mathbf{s}(w) = \sum_{u \in \boldsymbol{\ell}(E)} c_u u, \qquad \text{where} \qquad c_u = \sum_{\substack{w\text{-splits} \\ u = u^{(1)} \cdots u^{(k)}}} (-1)^k.$$

On the other hand, if u contains any two letters in order, then toggling the separator between them is an involution that changes the sign of w. So almost everything cancels, and

$$\mathbf{s}(w) = (-1)^{|w|} w^{\text{rev}}.$$

Alternatively: $c_u = (-1)^{|w|}$ times the reduced Euler characteristic of the simplex whose vertices are the possible separators between correctly ordered pairs of letters in u.

Duals of Hopf Monoids

Let $(\mathbf{H}, \mu, \Delta)$ be a vector Hopf monoid over $\Bbbk$. Its **dual** is the vector species

$$\mathbf{H}^*[I] = \mathrm{Hom}(\mathbf{H}[I], \Bbbk)$$

made into a Hopf monoid by

$$\Delta^*(\phi) = \phi \circ \mu \qquad \mu^*(\psi) = \psi \circ \Delta$$

$$\begin{tikzcd} \mathbf{H}[I] \otimes \mathbf{H}[J] \arrow[r, "\mu_{I,J}"] \arrow[dr, "\Delta^*(\phi)"'] & \mathbf{H}[I \sqcup J] \arrow[d, "\phi"] \\ & \Bbbk \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H}[I \sqcup J] \arrow[r, "\Delta_{I,J}"] \arrow[dr, "\mu^*(\psi)"'] & \mathbf{H}[I] \otimes \mathbf{H}[J] \arrow[d, "\psi"] \\ & \Bbbk \end{tikzcd}$$

Concretely: $\mu^* = \Delta^T$, $\Delta^* = \mu^T$ (w/r/t std. basis)
Also, $\mathbf{s}^* = \mathbf{s}$

The Hopf Monoid $\mathbf{L}^*$

As a vector species:

$$\mathbf{L}^*[I] = \mathbf{L}[I] = \Bbbk\boldsymbol{\ell}[I] = \Bbbk\{\text{linear orders on } I\}$$

As a Hopf monoid:

$$\mu^*_{I,J}(u \otimes v) = \sum_{\substack{w \in \boldsymbol{\ell}[I \sqcup J] \\ w|_I = u,\ w|_J = v}} w = \sum_{w \in \mathrm{Shuffle}(u,v)} w$$

Example

$$14 * 32 = 1432 + 1342 + 1324 + 3142 + 3124 + 3214 = 32 * 14$$

The Hopf Monoid $\mathbf{L}^*$

$$\Delta^*_{I,J}(w) = \sum_{\substack{u \in \boldsymbol{\ell}[I] \\ v \in \boldsymbol{\ell}[J] \\ u \cdot v = w}} u \otimes v$$

$$= \begin{cases} w|_I \otimes w|_J & \text{if } w|_I \text{ is an initial segment} \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathbf{L}$ is cocommutative but not commutative
- $\mathbf{L}^*$ is commutative but not cocommutative

Hadamard Products of Hopf Monoids

The **Hadamard product** $\mathbf{H} \times \mathbf{J}$ of Hopf monoids $\mathbf{H}, \mathbf{J}$ is defined by

$$(\mathbf{H} \times \mathbf{J})[I] = \mathbf{H}[I] \otimes \mathbf{J}[I], \quad \Delta^{\mathbf{H}\times\mathbf{J}} = \Delta^{\mathbf{H}} \otimes \Delta^{\mathbf{J}}, \quad \mu^{\mathbf{H}\times\mathbf{J}} = \mu^{\mathbf{H}} \otimes \mu^{\mathbf{J}}.$$

If the basis elements of $\mathbf{H}[I]$ are widgets on I, then the basis elements of $\mathbf{L} \times \mathbf{H}$ or $\mathbf{L}^* \times \mathbf{H}$ are *ordered* widgets.

Note: $\times$ is *not* multiplicative on antipodes — there is (probably) no formula giving $\mathbf{s}^{\mathbf{H}\times\mathbf{J}}$ in terms of $\mathbf{s}^{\mathbf{H}}$ and $\mathbf{s}^{\mathbf{J}}$.

Problem

What are the antipodes in $\mathbf{L} \times \mathbf{L}$, $\mathbf{L} \times \mathbf{L}^*$, $\mathbf{L}^* \times \mathbf{L}^*$, $\mathbf{L} \times \mathbf{L} \times \mathbf{L}$, *etc.?*

Normal Fans of Polytopes

Let $\mathfrak{p} \subseteq \mathbb{R}^n$ be a polytope and $\phi \in (\mathbb{R}^n)^*$. I.e., ϕ is a linear functional $\mathbb{R}^n \to \mathbb{R}$, say $\phi(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$.

Then $\mathfrak{p}_\phi = \{\mathbf{x} \in \mathfrak{p} : \ \phi(\mathbf{x}) \geqslant \phi(\mathbf{y}) \ \forall \mathbf{y} \in \mathfrak{p}\}$ is a face of $\mathfrak{p}$.

The **normal cone** of a face $\mathfrak{q} \subset \mathfrak{p}$ is

$$N(\mathfrak{q}) = N_{\mathfrak{p}}(\mathfrak{q}) = \{\phi \in (\mathbb{R}^n)^* : \ \mathfrak{p}_\phi = \mathfrak{q}\}.$$

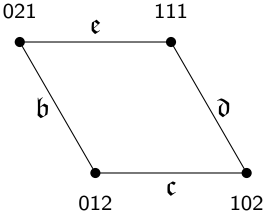
- $\dim N(\mathfrak{q}) = n - \dim \mathfrak{q}$
- $\overline{\mathfrak{q}} \subseteq \overline{\mathfrak{r}} \iff \overline{N(\mathfrak{r})} \subseteq \overline{N(\mathfrak{q})}$

The normal cones of all faces make up the **normal fan** $\mathcal{N}_{\mathfrak{p}}$.

If $\mathfrak{p}$ is an unbounded polyhedron then $\mathcal{N}_{\mathfrak{p}}$ is not complete (it only includes directions that define a face)

Normal Fans of Polytopes: Example

Let $\mathfrak{p} = \mathrm{conv}((0,1,2),(0,2,1),(1,1,1),(1,0,2)) \subset \mathbb{R}^3$.



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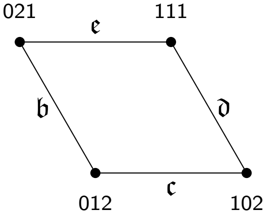

Define $\phi \in (\mathbb{R}^3)^*$ by $\phi(x_1, x_2, x_3) = \alpha_1 x_1 = \alpha_2 x_2 + \alpha_3 x_3$.

$\mathfrak{q}$	$N(\mathfrak{q})$	$\mathfrak{q}$	$N(\mathfrak{q})$
021	$\alpha_1, \alpha_3 < \alpha_2$	$\mathfrak{b}$	$\alpha_1 < \alpha_2 = \alpha_3$
012	$\alpha_1 < \alpha_2 < \alpha_3$	$\mathfrak{c}$	$\alpha_2 < \alpha_1 = \alpha_3$
102	$\alpha_2 < \alpha_3, \alpha_1$	$\mathfrak{d}$	$\alpha_2 = \alpha_3 < \alpha_1$
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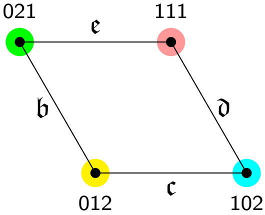

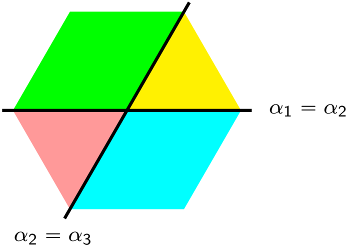

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The Standard Permutahedron

The **standard permutahedron** is the polytope $\Pi_n \subset \mathbb{R}^n$ whose vertices are the $n!$ permutations of $(1, 2, \ldots, n)$.

For a linear functional $\phi(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, the face of Π_n maximized by ϕ is determined by the relative order of $a_1, \ldots, a_n$.

That is, the normal fan $\mathcal{N}_{\Pi_n}$ is the **braid fan.**

The Braid Fan

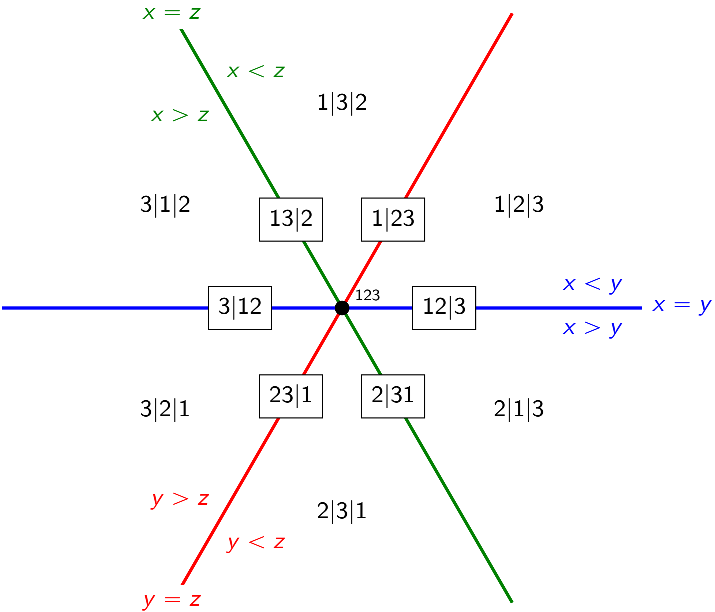
- The **braid arrangement** Br_n consists of the hyperplanes $H_{ij}: \ x_i = x_j$ in $\mathbb{R}^n$, for $1 \leqslant i < j \leqslant n$
- Br_n splits $\mathbb{R}^n$ into **faces** (on/above/below each H_{ij}).

  Each face is an open cone (closed under multiplication by positive scalars)
- Faces are indexed by **set compositions** of $[n]$:

$$x_2 = x_5 < x_3 < x_1 = x_6 < x_4 \qquad \Longleftrightarrow \qquad 25|3|16|4$$

- The collection of all faces is the **braid fan**.

The Braid Fan



The Braid Fan

Intersecting the braid fan with the unit sphere gives a simplicial $(n-2)$-sphere $\Sigma_n = \partial(\Pi_n^*)$. Each $A \models I$ corresponds to a face of Σ_n of dimension $|A| - 2$.

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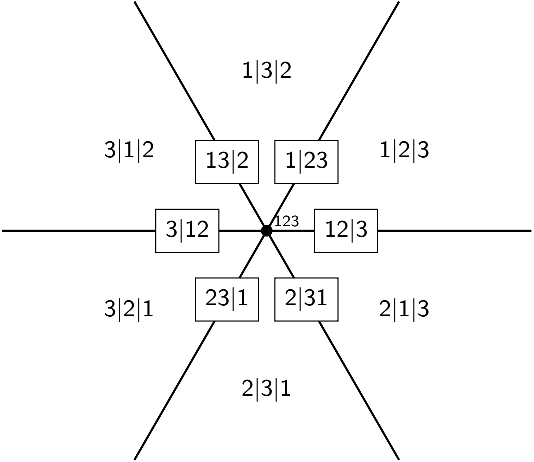

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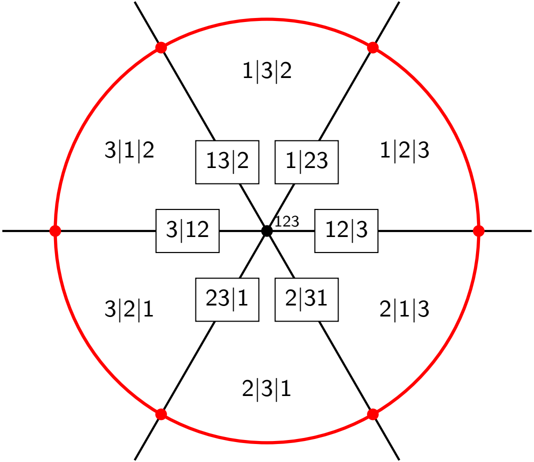
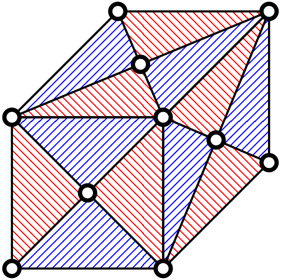


The Braid Fan

Here's (the front side of) Σ_4.



Advantage of working with Σ_n: we can import topological tools from the theory of simplicial complexes (like Euler characteristic) in working with set compositions (which arise naturally in Hopf-monoid land, e.g., in Takeuchi's formula)

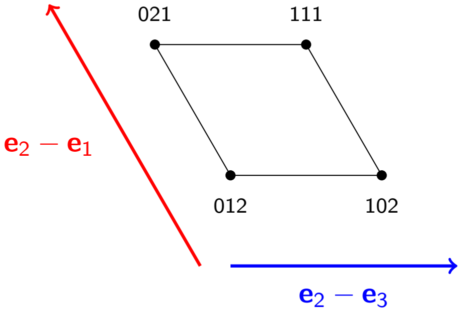
Generalized Permutahedra

A polyhedron $\mathfrak{p} \subseteq \mathbb{R}^n$ is a **generalized permutahedron** if the following equivalent conditions hold:

1. Every edge of $\mathfrak{p}$ is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some i, j
2. The normal fan of $\mathfrak{p}$ is a coarsening of [a subfan of] the braid fan
3. For any linear functional ϕ given by $\phi(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, the face of $\mathfrak{p}$ maximized by ϕ depends only on the relative order of $a_1, \ldots, a_n$.

Generalized Permutahedra: Example

Let $\mathfrak{p} = \text{conv}((0,1,2),(0,2,1),(1,1,1),(1,0,2)) \subset \mathbb{R}^3$.



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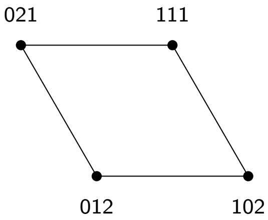

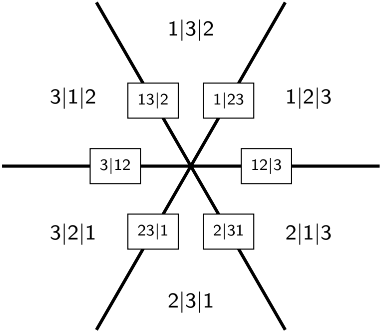

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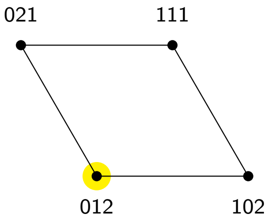

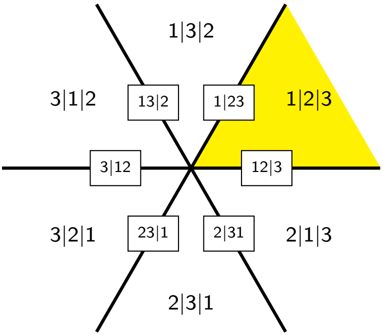

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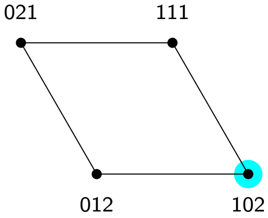

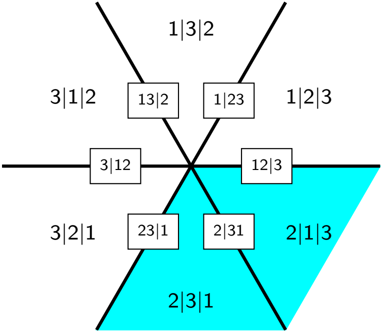

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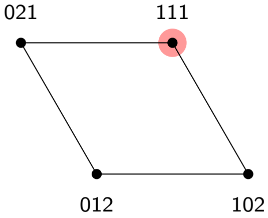

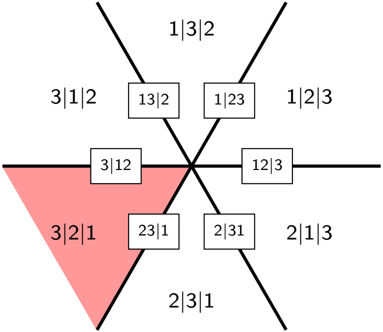

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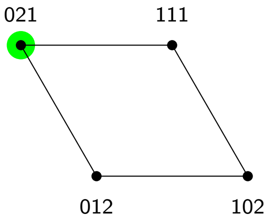

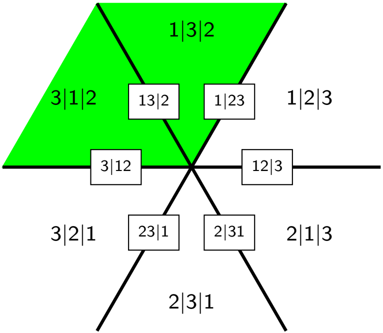

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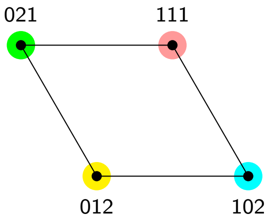

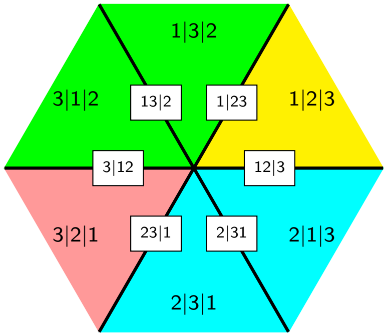

Matroid Base Polytopes

Let M be a matroid on ground set $[n]$ with basis system $\mathcal{B}$. The **matroid base polytope** is

$$\mathfrak{p}_M = \mathrm{conv}\{\chi_B : \ B \in \mathcal{B}\} \subset \mathbb{R}^n$$

where χ_B is the characteristic vector of B:

$$(\chi_B)_i = \begin{cases} 1 & \text{if } i \in B, \\ 0 & \text{if } i \notin B. \end{cases}$$

Example

$M =$ matroid rep'd by $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ $\qquad \mathcal{B} = \{12, 13, 23, 24, 34\}$

$\mathfrak{p}_M = \mathrm{conv}(1100, 1010, 0110, 0101, 0011) \subset \mathbb{R}^4$

Matroid Base Polytopes

Theorem

Matroid base polytopes are generalized permutahedra.

Proof idea #1:
Edges correspond to matroid basis exchanges.

Proof idea #2:
A linear functional $\phi(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ on $\mathfrak{p}_M$ can be maximized by running Kruskal's algorithm using $\mathbf{a}$ as a weight function. The algorithm only compares coefficients of $\mathbf{a}$ (no arithmetic).

Theorem (Gelfand–Goresky–Macpherson-Serganova)

Matroid base polytopes are exactly the bounded generalized permutahedra with 0/1 coordinates.

The Hopf Monoid of Generalized Permutahedra

Consider the vector species

$$\mathbf{GP}[I] = \Bbbk\{\text{bounded generalized permutahedra } \mathfrak{p} \subset \mathbb{R}^I\}.$$

Fact

Let $\mathfrak{p} \in \mathbf{GP}[I]$ and $I = J \sqcup K$. Let $1_J(x) = \sum_{j \in J} x_j$.
Then $\mathfrak{p}_{1_J} = \mathfrak{p}|J \times \mathfrak{p}/J$ for some $\mathfrak{p}|J \in \mathbf{GP}[J]$ and $\mathfrak{p}/J \in \mathbf{GP}[K]$.

(In other words, no equation defining the face minimized by $\mathbf{1}_J$ involves coordinates from both J and K.)

Theorem

$\mathbf{GP}$ *is a (commutative) Hopf monoid under the operations*

$$\mu(\mathfrak{p} \otimes \mathfrak{q}) = \mathfrak{p} \times \mathfrak{q} \qquad \Delta_{J,K}(\mathfrak{p}) = \mathfrak{p}|J \otimes \mathfrak{p}/J.$$

The Fact above implies $\mu_A(\Delta_A(\mathfrak{p})) = \mathfrak{p}_A$ for any $A \vDash I$.
This is relevant to the Takeuchi antipode formula.

The Hopf Monoid of Generalized Permutahedra

- Matroid base polytopes form a submonoid $\mathbf{Mat} \subset \mathbf{GP}$
- Allowing unbounded ("extended") generalized permutahedra produces an extension $\mathbf{GP}_+$
- Identifying (extended) generalized permutahedra with the same normal fans produces quotients $\overline{\mathbf{GP}}$, $\overline{\mathbf{GP}_+}$

The Antipode in GP

Theorem (Aguiar–Ardila)

Let $\mathfrak{p} \in \mathbf{GP}_+[I]$ *and* $n = |I|$. *Then*

$$\mathbf{s}(\mathfrak{p}) = \sum_{\textit{faces } \mathfrak{q} \subseteq \mathfrak{p}} (-1)^{n - \dim \mathfrak{q}} \, \mathfrak{q}.$$

Proof sketch: (1) Expand $\mathbf{s}(\mathfrak{p})$ using Takeuchi; collect like terms:

$$\mathbf{s}(\mathfrak{p}) = \sum_{A \models I} (-1)^{|A|} \mu_A(\Delta_A(\mathfrak{p})) = \sum_{A \models I} (-1)^{|A|} \mathfrak{p}_A = \sum_{\mathfrak{q} \subset \mathfrak{p}} \mathfrak{q} \underbrace{\sum_{C \in \mathcal{C}_{\mathfrak{q}}} (-1)^{\dim C}}_{\alpha_{\mathfrak{q}}}$$

where $\mathcal{C}_{\mathfrak{q}}$ is the set of braid faces in $N_{\mathfrak{p}}(\mathfrak{q})$

(2) Identify $\mathcal{C}_{\mathfrak{q}}$ with a $(\mathbb{B}^{n - \dim \mathfrak{q} - 2})^{\circ}$ open subcomplex of Σ_n.
Interpret $\alpha_{\mathfrak{q}}$ in terms of Euler characteristics of balls and spheres.

The Antipode in $\mathbf{L}^* \times \mathbf{GP}$

(Joint work with Federico Castillo and José Samper)

Idea: "If the basis elements of $\mathbf{H}[I]$ are widgets on I, then the basis elements of $\mathbf{L} \times \mathbf{H}$ or $\mathbf{L}^* \times \mathbf{H}$ are *ordered* widgets."

- $\mathbf{L}^* \times \mathbf{Mat}$: matroids with ordered ground set
- $\mathbf{L}^* \times \mathbf{GP}$: gen perms with ordered coordinates

Why bother to order the ground set?

- To study matroids using Hopf methods, we can study **Mat**.
- But to study related structures (shifted complexes, broken-circuit complexes, 0/1-GPs), we need an ordering
- These things do not give Hopf monoid extensions of **Mat**, but they do extend $\mathbf{L}^* \times \mathbf{Mat}$ (not $\mathbf{L} \times \mathbf{Mat}$!)

Example: 0/1-GPs

Recall that a matroid polytope is just a bounded generalized permutahedron with 0/1 coordinates.

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What about possibly-unbounded 0/1-GPs? These give rise to a larger family than just matroid complexes.

A complex Γ on I is a matroid iff $\Gamma|_A$ is pure for every $A \subseteq I$. This enables us to define

$$\Delta_{A,B}(\Gamma) = \Delta|A \otimes \Delta/A$$

where

$$\Delta/A = \mathrm{link}_\Gamma(\phi) = \{\sigma \subset B : \ \sigma \cup \phi \in \Delta\}$$

for **any** facet ϕ of $\Delta|A$.

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This breaks down if Γ is not a matroid!

Hopf monoids of EGPs

We don't want to work with $\mathbf{L}^* \times \mathbf{GP}_+$; we only want to consider orderings that are bounded w/r/t $\mathfrak{p}$ in the following sense.

Let $\mathfrak{p} \in \mathbf{GP}_+[I]$, i.e., $\mathfrak{p} \subset \mathbb{R}^I$ is a (possibly unbounded) GP. Define

$$\begin{aligned} \boldsymbol{\ell}_{\mathfrak{p}}[I] &= \{w \in \boldsymbol{\ell}[I] : \ \sigma_{w^{\mathrm{rev}}} \subseteq \mathcal{N}_{\mathfrak{p}}\} \\ &= \{w \in \boldsymbol{\ell}[I] : \ \mathfrak{p}_{w^{\mathrm{rev}}} \text{ is a well-defined vertex of } \mathfrak{p}\}. \end{aligned}$$

Now define a vector subspecies of $\mathbf{L}^* \times \mathbf{GP}_+$ by

$$\mathbf{OGP}_+[I] = \langle w \otimes \mathfrak{p} : \ \mathfrak{p} \in \mathbf{GP}_+[I], \ w \in \boldsymbol{\ell}_{\mathfrak{p}}[I] \rangle .$$

Theorem

$\mathbf{OGP}_+$ *is a Hopf monoid.*