THE INCIDENCE HOPF ALGEBRA OF GRAPHS

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Abstract. The *graph algebra* is a commutative, cocommutative, graded, connected incidence Hopf algebra, whose basis elements correspond to finite graphs and whose Hopf product and coproduct admit simple combinatorial descriptions. We give a new formula for the antipode in the graph algebra in terms of acyclic orientations; our formula contains many fewer terms than Takeuchi's and Schmitt's more general formulas for the antipode in an incidence Hopf algebra. Applications include several formulas (some old and some new) for evaluations of the Tutte polynomial.

1. Introduction

The *graph algebra* $\mathcal{G}$ is a commutative, cocommutative, graded, connected Hopf algebra, whose basis elements correspond to finite graphs, and whose Hopf product and coproduct admit simple combinatorial descriptions. The graph algebra was first considered by Schmitt in the context of incidence Hopf algebras [Sch94, §12] and furnishes an important example in the work of Aguiar, Bergeron and Sottile [ABS06, Example 4.5].

In this paper, we derive a nonrecursive formula (Theorem 3.1) for the Hopf antipode in $\mathcal{G}$. Our formula is specific to the graph algebra in that it involves acyclic orientations. Therefore, it is not merely a specialization of the antipode formulas of Takeuchi [Tak71] or Schmitt [Sch94] (in the more general settings of, respectively, connected bialgebras and incidence Hopf algebras). Aguiar and Ardila [AA] have independently discovered a more general antipode formula than ours, in the context of Hopf monoids; their work will appear in a forthcoming paper.

Our formula turns out to be well suited for studying polynomial graph invariants, including the Tutte polynomial $T_G(x, y)$ (see [BO92]) and various specializations of it. Specifically, to every graph G and character ζ on the graph algebra, we associate the function $P_{\zeta,G}(k)$ whose value at an integer k is $\zeta^k(G)$, where the superscript denotes convolution power. For example, if ζ is the characteristic function of edgeless graphs, then $P_{\zeta,G}(k)$ is the chromatic polynomial of G. In fact, it turns out that $P_{\zeta,G}(k)$ is a polynomial

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function of k for *all* characters ζ, so we may regard P_ζ as a map $\mathcal{G} \to \mathbb{C}[k]$ sending G to $P_{\zeta,G}(k)$. This map is in fact a morphism of Hopf algebras; it is not graded (so it is not quite a morphism of *combinatorial* Hopf algebras in the sense of Aguiar, Bergeron and Sottile [ABS06]) but does preserve the canonical filtration by degree. Together with the antipode formula, this observation leads to combinatorial interpretations of the convolution inverses of several natural characters, as we discuss in Section 3.1.

The Tutte polynomial $T_G(x, y)$ can itself be viewed as a character on the graph algebra. We prove that its k-th convolution power itself is a Tutte evaluation at rational functions in x, y, k (Theorem 4.1). This result implies several well-known formulas such as Stanley's formula for acyclic orientations in terms of the chromatic polynomial [Sta73], as well as some interpretations of less familiar specializations of the Tutte polynomial, and an unusual-looking reciprocity relation between complete graphs of different sizes (Proposition 5.1 and Corollary 5.2).

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2. Hopf algebras

2.1. **Basic definitions.** We briefly review the basic facts about Hopf algebras, omitting most of the proofs. Good sources for the full details include Sweedler [Swe69] and (for combinatorial Hopf algebras) Aguiar, Bergeron and Sottile [ABS06]. For the more general setting of Hopf monoids, see Aguiar and Mahajan [AM10]. We do not know of specific references for Lemma 2.1 and Proposition 2.2, but they are well known as part of the general folklore of (combinatorial) Hopf algebras.

Fix a field $\mathbb{F}$ of characteristic 0 (typically $\mathbb{F} = \mathbb{C}$). A *bialgebra* $\mathcal{H}$ is a vector space over $\mathbb{F}$ equipped with linear maps

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \qquad u : \mathbb{F} \to \mathcal{H}, \qquad \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad \epsilon : \mathcal{H} \to \mathbb{F},$$

respectively the *multiplication*, *unit*, *comultiplication*, and *counit*, such that the following properties are satisfied:

(1) $m \circ (m \otimes I) = m \circ (I \otimes m)$ (associativity);
(2) $m \circ (u \otimes I) = m \circ (I \otimes u) = I$ (where I is the identity map on $\mathcal{H}$);
(3) $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$ (coassociativity);
(4) $(\epsilon \otimes I) \circ \Delta = (I \otimes \epsilon) \circ \Delta = I$; and
(5) Δ and ϵ are multiplicative (equivalently, m and u are comultiplicative).

If there exists a bialgebra automorphism $S : \mathcal{H} \to \mathcal{H}$ such that $m \circ (S \otimes I) \circ \Delta = m \circ (I \otimes S) \circ \Delta = u \circ \epsilon$, then $\mathcal{H}$ is a *Hopf algebra* and S is its *antipode*. It can be shown that S is the unique automorphism of $\mathcal{H}$ with this property.

It is often convenient to write expressions such as coproducts in *Sweedler notation*, where the index of summation is suppressed: for instance, $\Delta(h) = \sum h_1 \otimes h_2$ rather than $\Delta(h) = \sum_i h_1^{(i)} \otimes h_2^{(i)}$.

A Hopf algebra $\mathcal{H}$ is *graded* if $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ as vector spaces, and multiplication and comultiplication respect this decomposition, i.e.,

$$m(\mathcal{H}_i \otimes \mathcal{H}_j) \subseteq \mathcal{H}_{i+j} \quad \text{and} \quad \Delta(\mathcal{H}_n) \subseteq \sum_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j.$$

If $h \in \mathcal{H}_i$, we say that h is *homogeneous of degree* i. The algebra $\mathcal{H}$ is *connected* if $\dim(\mathcal{H}_0) = 1$. Most Hopf algebras arising naturally in combinatorics are graded and connected, and every algebra we consider henceforth will be assumed to have these properties.

Let $\mathcal{H}$ be a graded and connected bialgebra. There is a unique Hopf antipode on $\mathcal{H}$, defined inductively by the formulas

$$S(h) = h \qquad \text{for } h \in \mathcal{H}_0, \tag{1a}$$

$$(m \circ (I \otimes S) \circ \Delta)(h) = 0 \qquad \text{for } h \in \mathcal{H}_i,\ i > 0. \tag{1b}$$

Formula (1b) can be rewritten more explicitly using Sweedler notation. If $\Delta(h) = \sum h_1 \otimes h_2$, then $\sum h_1 S(h_2) = 0$, so solving for $S(h)$ gives

$$S(h) = -\sum h_1 S(h_2), \tag{2}$$

the sum over all summands in which the degree of h_2 is strictly less than that of h.

A *character* of a Hopf algebra $\mathcal{H}$ is a multiplicative linear map $\phi : \mathcal{H} \to \mathbb{F}$. The *convolution product* of two characters is $\phi * \psi = (\phi \otimes \psi) \circ \Delta$. That is, if $\Delta h = \sum h_1 \otimes h_2$, then

$$(\phi * \psi)(h) = \sum \phi(h_1)\psi(h_2)$$

with both sums in Sweedler notation. We write ϕ^k for the k-th convolution power of ϕ; if $k < 0$ then $\phi^k = (\phi^{-1})^{-k}$. Convolution makes the set of characters $\mathbb{X}(\mathcal{H})$ into a group, with identity ϵ and inverse given by

$$\phi^{-1} = \phi \circ S. \tag{3}$$

There is a natural involutive automorphism $\phi \mapsto \overline{\phi}$ of $\mathbb{X}(\mathcal{H})$, given by $\overline{\phi}(h) = (-1)^n \phi(h)$ for $h \in \mathcal{H}_n$. If $\mathcal{H}$ is a graded connected Hopf algebra and $\zeta \in \mathbb{X}(H)$, then the pair $(\mathcal{H}, \zeta)$ is called a *combinatorial Hopf algebra*, or CHA for short. A *morphism* of CHAs $\Phi : (\mathcal{H}, \zeta) \to (\mathcal{H}', \zeta')$ is a linear transformation $\mathcal{H} \to \mathcal{H}'$ that is a morphism of Hopf algebras (i.e., a linear transformation that preserves the operations of a bialgebra) such that $\zeta \circ \Phi = \zeta'$.

2.2. **The binomial Hopf algebra.** The *binomial Hopf algebra* is the ring of polynomials $\mathbb{F}[k]$ in one variable k, with the usual multiplicative structure; comultiplication defined by $\Delta(f(k)) = f(k \otimes 1 + 1 \otimes k)$ and $\Delta(1) = 1 \otimes 1$; counit $\epsilon(f(k)) = \epsilon_0(f(k)) = f(0)$; and character $\epsilon_1(f(k)) = f(1)$. A theme of this article, that polynomial invariants of elements of a Hopf algebra $\mathcal{H}$

can be viewed as the values of a morphism $\mathcal{H} \to \mathbb{F}[k]$. The main result in this vein, Proposition 2.2, can be proved with elementary methods, but we instead give a longer proof that illustrates the connection to the work of Aguiar, Bergeron, and Sottile [ABS06]. In order to do so, we begin by reviewing some facts about compositions and quasisymmetric functions; for more details, see, e.g., [Sta99, §7.19].

Let n be a nonnegative integer. A *composition of* n is an ordered sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of positive integers such that $\alpha_1 + \cdots + \alpha_\ell = n$; in this case we write $\alpha \vDash n$. The number $\ell = \ell(\alpha)$ is the *length* of α. The corresponding *monomial quasisymmetric function* is the formal power series

$$M_\alpha = \sum_{0<i_1<\cdots<i_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell} \tag{4}$$

in countably infinitely many commuting variables $\{x_1, x_2, \dots\}$. The $\mathbb{F}$-vector space spanned by the M_α is denoted $\mathcal{QSym}$. This is in fact a Hopf algebra, with the natural addition, multiplication, and unit; counit

$$\epsilon(M_\alpha) = \begin{cases} 1 & \text{if } \ell(\alpha) = 0, \\ 0 & \text{if } \ell(\alpha) > 0; \end{cases}$$

and comultiplication

$$\Delta(M_{(\alpha_1,\dots,\alpha_\ell)}) = \sum_{i=0}^{\ell} M_{(\alpha_1,\dots,\alpha_i)} M_{(\alpha_{i+1},\dots,\alpha_\ell)}.$$

For $F(x_1, x_2, \dots) \in \mathcal{QSym}$, let $\zeta_Q(F)$ be the number obtained by substituting $x_1 = 1$ and $x_2 = x_3 = \cdots = 0$. The map ζ_Q is a character on $\mathcal{QSym}$. Aguiar, Bergeron, and Sottile [ABS06, Thm. 4.1] proved that $(\mathcal{QSym}, \zeta_Q)$ is a terminal object in the category of CHAs, i.e., that every CHA $(\mathcal{H}, \zeta)$ has a unique morphism

$$\Psi : (\mathcal{H}, \zeta) \to (\mathcal{QSym}, \zeta_Q),$$

given explicitly on $h \in \mathcal{H}_n$ by

$$\Psi(h) = \sum_{\alpha \vDash n} \zeta_\alpha(h) M_\alpha\,;$$

here $\zeta_\alpha : \mathcal{H} \to \mathbb{F}$ is the composite function

$$\mathcal{H} \xrightarrow{\Delta^{\ell-1}} \mathcal{H}^{\otimes \ell} \xrightarrow{\pi_\alpha} \mathcal{H}_{\alpha_1} \otimes \cdots \otimes \mathcal{H}_{\alpha_\ell} \xrightarrow{\zeta^{\otimes \ell}} \mathbb{F}$$

where $\ell = \ell(\alpha)$ is the number of parts of α, and π_α is the tensor product of the canonical projections of $\mathcal{H}$ onto the graded pieces $\mathcal{H}_{\alpha_i}$.

For $F(x_1, x_2, \dots) \in \mathcal{QSym}$, let $\mathrm{ps}_k^1(F)$ be the number obtained by substituting $x_1 = \cdots = x_k = 1$ and $x_{k+1} = \cdots = 0$. In particular, $\mathrm{ps}_1^1 = \zeta_Q$. The map ps_k^1 is a specialization of a map called the *principal specialization* [Sta99, pp. 302–303]. By (4), we have

$$\mathrm{ps}_k^1(M_\alpha) = \frac{k(k-1)\cdots(k-\ell(\alpha)+1)}{\ell(\alpha)!} = \binom{k}{\ell(\alpha)}.$$

Accordingly, we can regard ps^1_k as a map

$$\Pi : \mathcal{Q}Sym \to \mathbb{F}[k]$$

sending M_α to $\mathrm{ps}^1_k(M_\alpha)$. (The reason for the apparently redundant notation is that when we write ps^1_k, we are regarding k as an integer, while when we write Π, we are regarding k as the indeterminate in the polynomial ring $\mathbb{F}[k]$.)

Lemma 2.1. *The map $\Pi : \mathcal{Q}Sym \to \mathbb{F}[k]$ is a morphism of Hopf algebras. Moreover, $\zeta_Q = \epsilon_1 \circ \Pi$.*

We remark that Π is not a morphism of *combinatorial* Hopf algebras because it is not graded (i.e., $\Pi(M_\alpha)$ is not homogeneous), merely filtered by degree.

Proof. The definition of ps^1_k implies that Π is a homomorphism of $\mathbb{F}$-algebras. To see that it is in fact a Hopf morphism, we must show that $(\Pi \otimes \Pi) \circ \Delta = \Delta \circ \Pi$. It suffices to check this for the basis $\{M_\alpha\}$. Let $x = k \otimes 1$ and $y = 1 \otimes k$; then

$$\begin{aligned}(\Pi \otimes \Pi)(\Delta M_\alpha) &= (\Pi \otimes \Pi)\left(\sum_{j=0}^{\ell} M_{(\alpha_1,\dots,\alpha_j)} \otimes M_{(\alpha_{j+1},\dots,\alpha_\ell)}\right)\\ &= \sum_{j=0}^{\ell} \binom{x}{j}\binom{y}{\ell-j} = \binom{x+y}{\ell} = \Delta\binom{k}{\ell} = \Delta(\Pi(M_\alpha)).\end{aligned}$$

(The third equality is a standard identity of binomial coefficients [Sta97, Ex. 1.1.17] that holds for all nonnegative integers x, y; therefore, it is an identity of polynomials.)

For the second assertion of the lemma, we have

$$\zeta_Q(M_\alpha) = \begin{cases} 1 & \text{if } \ell(\alpha) \le 1 \\ 0 & \text{if } \ell(\alpha) > 1 \end{cases} = \left.\frac{k(k-1)\cdots(k-\ell+1)}{\ell!}\right|_{k=1} = \epsilon_1(\Pi(M_\alpha)).$$

$\square$

We now come to the main result of this section. Again, this fact is not new, but is part of the folklore of (combinatorial) Hopf algebras.

Proposition 2.2 (Polynomiality)**.** *For every combinatorial Hopf algebra $(\mathcal{H}, \zeta)$, there is a CHA morphism*

$$P_\zeta : (\mathcal{H}, \zeta) \to (\mathbb{F}[k], \epsilon_1)$$

mapping h to the unique polynomial $P_{\zeta,h}(k)$ such that

$$P_{\zeta,h}(k) = \zeta^k(h) \qquad \forall k \in \mathbb{Z}.$$

Moreover, if $h \in \mathcal{H}_n$, then $P_{\zeta,h}(k)$ is a polynomial in k of degree at most n.

Proof. We will show that $P_{\zeta,h}(k) = \Pi(\Psi(h))$ for all $h \in \mathcal{H}$. It is not hard to see that P_ζ is a vector space homomorphism, so it is sufficient to consider the case that h is homogeneous of degree n. The desired equality follows from the calculation

$$\begin{aligned} P_{\zeta,h}(k) \;=\; \zeta^k(h) \;&=\; \sum \zeta(h_1)\cdots\zeta(h_k) && \text{(5a)}\\ &=\; \sum_{\alpha \vDash n} \binom{k}{\ell(\alpha)} \zeta_\alpha(h) && \text{(5b)}\\ &=\; \Pi \sum_{\alpha \vDash n} \zeta_\alpha(h) M_\alpha \;=\; \Pi(\Psi(h)). \end{aligned}$$

The sum in (5a) is Sweedler notation. The only tricky equality is (5b); for this, note that each summand $\zeta(h_1)\cdots\zeta(h_k)$ in (5a) arises from an ordered list $(h_1,\dots,h_k)$ of homogeneous elements of $\mathcal{H}$ whose degrees sum to n. Define the *essence* of a summand $\zeta(h_1)\cdots\zeta(h_k)$ to be the sublist of $(h_1,\dots,h_k)$ consisting of elements of strictly positive degree. Each equivalence class of summands with the same essence $(h_{i_1},\dots,h_{i_\ell})$ contains precisely $\binom{k}{\ell}$ summands (since by the counit property, the positive-degree factors may occur in any positions) and thus contributes $\binom{k}{\ell}\zeta(h_{i_1})\cdots\zeta(h_{i_\ell})$ to the sum. Collecting together all equivalence classes whose essences have the same degree sequence α contributes $\binom{k}{\ell(\alpha)}\zeta_\alpha(h)$.

Finally, observe that $\binom{k}{\ell} = \frac{k(k-1)\cdots(k-\ell+1)}{\ell(\ell-1)\cdots 1}$ is a polynomial in k of degree ℓ, and that every composition $\alpha \vDash n$ has at most n parts, so $P_{\zeta,h}(k)$ is a polynomial in k of degree at most n. $\square$

One can also prove Proposition 2.2 by direct calculation, for instance, by showing that $D^{n+1}P_{\zeta,h}(k) = 0$, where D is the difference operator $DP(k) = P(k) - P(k-1)$.

Proposition 2.2 provides a way of translating characters on a Hopf algebra into polynomial invariants of its elements, just as the Aguiar–Bergeron–Sottile theorem translates characters into quasisymmetric-function invariants. Passing from quasisymmetric functions to polynomials may lose information, but may also lead to more explicit formulas.

2.3. **Graphs and the graph Hopf algebra.** We now describe the Hopf algebra that is the subject of this article. (The literature contains many other instances of Hopf algebras of graphs; for example, this is not the same Hopf structure as the algebra studied by Novelli, Thibon and Thiéry [NTT04].)

First, we set up graph-theoretic notation and terminology. The notation $G = (V,E)$ means that G is a finite, undirected graph with vertex set V and edge set E; we may then write $G_{V',E'}$ for the subgraph with vertex set V' and edge set E'. (We could also write simply (V',E'), but we often wish to emphasize that this graph is a subgraph of G.) Loops and multiple edges are allowed. The sets of vertices and edges of a graph G will be denoted $V(G)$ and $E(G)$ respectively; no confusion should arise from this apparent abuse of notation. The numbers of vertices, edges and connected components are denoted $n(G)$, $e(G)$, $c(G)$ respectively (or sometimes n, e, c). The induced subgraph on a vertex set $T \subseteq V$ will be denoted $G|_T$. The complement of T will be denoted $\overline{T}$. If S and T are vertex sets, then $[S, T]$ denotes the set of all edges with one endpoint in S and one endpoint in T. The complete graph on n vertices is written K_n; note that we permit the possibility $n = 0$.

The *rank* $\mathrm{rk}(F)$ of a subset $F \subseteq E(G)$ is the size of any maximal acyclic subset of F. Meanwhile, the set F is called a *flat* if, whenever the endpoints of an edge e are connected by a path in F, then $e \in F$. (These are precisely the flats of the graphic matroid of G.) Equivalently, F is a flat iff $\mathrm{rk}(F') > \mathrm{rk}(F)$ for every $F' \supsetneq F$.

For an edge $e \in E$, the *contraction* G/e is obtained by identifying the two endpoints of e (which is a trivial step if e is a loop) and then removing e. For an edge set $F \subseteq E$, the symbol G/F denotes the graph obtained by successively contracting every edge of F (the order does not matter). Observe that if F is a flat, then G/F contains no loops.

An *acyclic orientation* of G is a choice of orientation of all the edges that admits no directed cycles. Let

$$\begin{aligned} \mathcal{F}(G) &= \{\text{flats of } G\}, \\ \mathcal{A}(G) &= \{\text{acyclic orientations of } G\}, \\ a(G) &= |\mathcal{A}(G)|. \end{aligned}$$

Note that if G has one or more loops, then $a(G) = 0$; otherwise, the number of acyclic orientations is unchanged upon replacing G with its underlying simple graph.

Now we can define our central object of study. The *graph algebra* is the $\mathbb{F}$-vector space $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n$, where $\mathcal{G}_n$ is the linear span of isomorphism classes of graphs on n vertices. This is a graded connected Hopf algebra, with multiplication $m(G \otimes H) = G \cdot H = G \uplus H$ (where $\uplus$ denotes disjoint union); unit $u(1) = K_0$; comultiplication

$$\Delta(G) = \sum_{T \subseteq V(G)} G|_T \otimes G|_{\overline{T}},$$

and counit

$$\epsilon(G) = \begin{cases} 1 & \text{if } n(G) = 0, \\ 0 & \text{if } n(G) > 0. \end{cases}$$

The graph algebra is commutative and cocommutative; in particular, its character group $\mathbb{X}(G)$ is abelian. As proved by Schmitt [Sch94, eq. (12.1)], the antipode in $\mathcal{G}$ is given combinatorially by

$$S(G) = \sum_{\pi} (-1)^{|\pi|} |\pi|! \, G_\pi$$

where the sum runs over all ordered partitions π of $V(G)$ into nonempty sets (or "blocks"), and G_π is the disjoint union of the induced subgraphs on the blocks. This is a consequence of Takeuchi's more general formula for connected Hopf algebras [Tak71, Lemma 14]; see also [AM10, §2.3.3 and §8.4], [AS05, §5], [Mon93].

The graph algebra admits two canonical involutions on characters:

$$\overline{\phi}(G) = (-1)^{n(G)}\phi(G), \qquad \tilde{\phi}(G) = (-1)^{\mathrm{rk}(G)}\phi(G),$$

where $\mathrm{rk}(G)$ denotes the graph rank of G (that is, the number of edges in a spanning tree). As always, $\phi \mapsto \overline{\phi}$ is an automorphism of $\mathbb{X}(\mathcal{G})$; on the other hand, $\phi \mapsto \tilde{\phi}$ is not. The graph algebra was studied by Schmitt [Sch94] and appears as the *chromatic algebra* in the work of Aguiar, Bergeron and Sottile [ABS06], where it is equipped with the character

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges,} \\ 0 & \text{if } G \text{ has at least one edge.} \end{cases} \tag{6}$$

We will study several characters on $\mathcal{G}$ other than ζ.

3. A combinatorial antipode formula

In this section, we prove a new combinatorial formula for the Hopf antipode in $\mathcal{G}$. Unlike Takeuchi's and Schmitt's formulas, our formula applies only to $\mathcal{G}$ and and does not generalize to other incidence algebras. On the other hand, our formula involves many fewer summands, which makes it useful for enumerative formulas involving characters. As noted in the introduction, Aguiar and Ardila have independently discovered a more general antipode formula in the context of Hopf monoids.

Theorem 3.1. *Let $G = (V, E)$ be a graph with $n = |V|$. Then*

$$S(G) = \sum_{F \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F)} a(G/F) G_{V,F}.$$

Proof. We proceed by induction on n. If G has no vertices, i.e., $G = K_0$, then $S(K_0) = K_0$ by (1a). Indeed, $\mathcal{F}(K_0) = \{\emptyset\}$, so the desired formula reduces to $S(K_0) = K_0$.

On the other hand, if G has at least one vertex, then by (2) we have

$$\begin{aligned} S(G) &= -\sum_{\emptyset \neq T \subseteq V} G|_T \cdot S(G|_{\overline{T}}) \\ &= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{F \in \mathcal{F}(G|_{\overline{T}})} (-1)^{n-|T|-\mathrm{rk}(F)} a(G|_{\overline{T}}/F) G_{\overline{T},F} \\ &= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{F \in \mathcal{F}(G|_{\overline{T}})} \sum_{\mathcal{O} \in \mathcal{A}(G|_{\overline{T}}/F)} (-1)^{n-|T|-\mathrm{rk}(F)} G_{\overline{T},F}. \end{aligned} \tag{7}$$

Now we establish a bijection which will allow us to interchange the order of summation.

First, suppose we are given a nonempty vertex set $T \subseteq V$, a flat F of $G|_{\overline{T}}$, and an acyclic orientation $\mathcal{O}$ of $G|_{\overline{T}}/F$. Let $F' = E(G|_T) \cup F$; this is a flat of G. Moreover, we can construct an acyclic orientation $\mathcal{O}'$ of G/F' by orienting all edges in $[\overline{T},\overline{T}]$ as in $\mathcal{O}$, and orienting all edges in $[T,\overline{T}]$ towards $\overline{T}$. Let $S_{\mathcal{O}'}$ be the set of sources of $\mathcal{O}'$ (that is, vertices with no in-edges); then the image T' of T under the contraction of F' is a nonempty subset of $S_{\mathcal{O}'}$.

Second, suppose we are given a flat F' of G, an acyclic orientation $\mathcal{O}'$ of G/F', and a set T' such that $\emptyset \neq T' \subseteq S_{\mathcal{O}'}$. Let T be the inverse image of T' under contraction of F'. Then $F = F' \setminus E(G|_T)$ is a flat of $G|_{\overline{T}}$, and we can construct an acyclic orientation $\mathcal{O}$ of $G|_{\overline{T}}/F$ by orienting all edges as in $\mathcal{O}'$.

It is straightforward to check that these constructions are inverses. Therefore, we have a bijection

$$\left\{ (T,F,\mathcal{O}) \;\middle|\; \begin{array}{l} \emptyset \neq T \subseteq V(G) \\ F \in \mathcal{F}(G|_{\overline{T}}) \\ \mathcal{O} \in \mathcal{A}(G|_{\overline{T}}/F) \end{array} \right\} \to \left\{ (F',\mathcal{O}',T') \;\middle|\; \begin{array}{l} F' \in \mathcal{F}(G) \\ \mathcal{O}' \in \mathcal{A}(G/F') \\ \emptyset \neq T' \subseteq S_{\mathcal{O}'} \end{array} \right\}$$

with the following properties:

- $|T'|$ is the number of components of $G|_T$;
- $|T|-|T'| = \mathrm{rk}(G|_T) = \mathrm{rk}(F') - \mathrm{rk}(F)$, so $|T|+\mathrm{rk}(F) = |T'|+\mathrm{rk}(F')$;
- $G|_T \cdot G_{\overline{T},F} = G_{V,F'}$ in the graph algebra $\mathcal{G}$.

Therefore, (7) gives

$$\begin{aligned} S(G) &= -\sum_{F' \in \mathcal{F}(G)} \sum_{\mathcal{O}' \in \mathcal{A}(G/F')} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{n-|T'|-\mathrm{rk}(F')} G_{V,F'} \\ &= -\sum_{F' \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F')} G_{V,F'} \sum_{\mathcal{O}' \in \mathcal{A}(G/F')} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{|T'|} \\ &= \sum_{F' \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F')} a(G/F') G_{V,F'}. \qquad \square \end{aligned}$$

3.1. **Inversion of characters.** We now apply the antipode formula to give combinatorial interpretations of several instances of inversion in the group of characters.

Proposition 3.2. *Let Ω be any family of graphs such that $G \uplus H \in \Omega$ if and only if $G \in \Omega$ and $H \in \Omega$; equivalently, such that the function*

$$\psi_\Omega(G) = \begin{cases} 1 & \text{if } G \in \Omega, \\ 0 & \text{if } G \notin \Omega \end{cases}$$

is a character. Then

$$\psi_\Omega^{-1}(G) = \sum_{F \in \mathcal{F}(G):\ G_{V,F} \in \Omega} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

Proof. From equation (3) and Theorem 3.1, we have

$$\begin{aligned}\psi_\Omega^{-1}(G) = \psi(SG) &= \sum_F (-1)^{n-\mathrm{rk}(F)} a(G/F)\psi(G_{V,F}) \\ &= \sum_{F\in\Omega} (-1)^{n-\mathrm{rk}(F)} a(G/F).\end{aligned}$$

□

Example 3.3. Let Ω be the family of graphs with no edges. Then ψ_Ω is just the character ζ of (6), and $P_{\zeta,G}(k)$ is the chromatic polynomial $\chi(G;k)$ of G. Therefore, Proposition 3.2 implies that $\psi_\Omega^{-1}(G) = \chi(G;-1) = (-1)^n a(G)$, a classic theorem of Stanley [Sta73].

Example 3.4. Let Ω be the family of acyclic graphs, and let $\alpha = \psi_\Omega$. Then

$$\alpha^{-1}(G) = \sum_{\text{acyclic flats } F} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

We examine two special cases. First, suppose that $G = C_n$, the cycle of length n. The acyclic flats of G are just the sets of $n-2$ or fewer edges, so an elementary calculation (which we omit) gives $\alpha^{-1}(C_n) = (-1)^n + 1$, the Euler characteristic of an n-sphere.

For many other families Ω, the Ω-free flats of C_n are just its flats, i.e., the edge sets of cardinality $\neq n-1$. In such cases, the same omitted calculation gives $\psi_\Omega^{-1}(C_n) = (-1)^n$.

Second, suppose that $G = K_n$. Now the acyclic flats of G are matchings, i.e., sets of edges that cover no vertex more than once. For $0 \leq m \leq \lfloor n/2 \rfloor$, the number of m-edge matchings is $n!/(2^m(n-2m)!m!)$, and contracting each such matching yields a graph whose underlying simple graph is K_{n-m}. Therefore

$$\alpha^{-1}(K_n) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^{n-m} \frac{n!}{2^m(n-2m)!m!}(n-m)!.$$

Starting at $n = 1$, these numbers are as follows:

$$-1,\ 1,\ 0,\ -6,\ 30,\ -90,\ 0,\ 2520,\ -22680,\ 113400,\ 0,\ -7484400,\ \ldots.$$

This is sequence A009775 in [Slo10], for which the exponential generating function is $-\tanh(\ln(1+x))$.

Example 3.5. Fix any connected graph H. Say that G is *H-free* if it has no subgraph isomorphic to H. (This is a stronger condition than saying that G has no *induced* subgraph isomorphic to H.) The corresponding *avoidance character* η_H is defined by

$$\eta_H(G) = \begin{cases} 1 & \text{if } G \text{ is } H\text{-free}, \\ 0 & \text{otherwise.}\end{cases}$$

Avoidance characters are special cases of the characters described by Proposition 3.2; specifically, $\eta_H = \psi_\Omega$, where Ω is the family of graphs with no subgraph isomorphic to H. For instance, $\eta_{K_1} = \epsilon$ and $\eta_{K_2} = \zeta$; more generally, if $H = K_{m,1}$ is the complete bipartite graph with partite sets of sizes m and 1, then the corresponding avoidance character η_H detects whether or not G has maximum degree strictly less than m. In general, for an avoidance character η_H, the summands in Proposition 3.2 include only the H-free flats.

In the case that T is a tree, every subset $F \subseteq E(T)$ is a flat, and every contraction T/F is acyclic, so all $2^{e(T/F)}$ orientations of T/F are acyclic. Therefore, Proposition 3.2 simplifies to

$$\eta_H^{-1}(T) \;=\; \sum_F (-1)^{r+1-|F|} 2^{r-|F|} \;=\; -\sum_F (-2)^{r-|F|}$$

where $r = r(T) = n(T)-1$, and both sums run over all H-free forests $F \subseteq T$.

Example 3.6. For every avoidance character η_H, the polynomial $P_{\eta_H}(G;k)$ counts the number of k-colorings of G such that every color-induced subgraph is H-free. As an extreme example, if $G = H$, then $P_{\eta_G}(G;k) = k^{n(G)} - k$, because the non-$G$-free colorings are precisely those using only one color.

If $H = K_{m,1}$, then $P_m(G;k) = P_{\eta_H}(G;k)$ counts the k-colorings such that no vertex belongs to m or more monochromatic edges, or equivalently such that no color-induced subgraph has a vertex of degree $\geq m$. We call this the *degree-chromatic polynomial*; if $m = 1$, then $P_1(G;k)$ is just the usual chromatic polynomial. In general, two trees with the same number of vertices need not have the same degree-chromatic polynomials for all m (though they do share the same chromatic polynomial). For example, if Z is the three-edge path on four vertices and $\mathsf{Y} = K_{3,1}$ is the three-edge star, then $P_2(\mathsf{Z};k) = k^4 - 2k^2 + k$ and $P_2(\mathsf{Y};k) = k^4 - 3k^2 + 2k$.

In an earlier version of this article, we had conjectured, based on experimental evidence, that if T is any tree on n vertices and $m < n$, then

$$P_m(T;k) = k^n - \sum_{v \in V(T)} \binom{d_T(v)}{m} k^{n-m} + \text{(lower order terms)}$$

where $d_T(v)$ denotes the degree of vertex v. This conjecture has since been proven combinatorially by Diego Cifuentes [Cif11].

4. Tutte characters

The *Tutte polynomial* $T_G(x,y)$ is a powerful graph invariant with many important properties (for a comprehensive survey, see [BO92]). It is defined in closed form by the formula

$$T_G(x,y) \;=\; \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}(G)-\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}$$

where $\mathrm{rk}(A)$ is the graph rank of A, and $\mathrm{null}(A) = |A| - \mathrm{rk}(A)$ (the *nullity* of A). The Tutte polynomial is a universal deletion-contraction invariant in the sense that every graph invariant satisfying a deletion-contraction recurrence can be obtained from $T_G(x,y)$ via a standard "recipe" [Bol98, p. 340]. In particular, $T_G(x,y)$ is multiplicative on connected components, so we can regard it as a character on the graph algebra:

$$\tau_{x,y}(G) = T_G(x,y).$$

We may regard x, y either as indeterminates or as (typically integer-valued) parameters. It is often more convenient to work with the *rank-nullity polynomial*

$$R_G(x,y) = \sum_{A\subseteq E} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)} = (x-1)^{\mathrm{rk}(G)} T_G(x/(x-1), y) \quad (8)$$

which carries the same information as $T_G(x,y)$, and is also multiplicative on connected components, hence is a character on $\mathcal{G}$. Note that $R_G(1,y) = 1$, and that

$$T_G(x,y) = (x-1)^{\mathrm{rk}(G)} R_G(x/(x-1), y). \quad (9)$$

Let $\rho_{x,y}$ denote the function $G \mapsto R_G(x,y)$, viewed as a character of the graph algebra $\mathcal{G}$.

For later use, we record the relationship between ρ and τ:

$$\tau_{x,y} = (x-1)^{\mathrm{rk}(G)} \rho_{x/(x-1),y}, \qquad \rho_{x,y} = (x-1)^{\mathrm{rk}(G)} \tau_{x/(x-1),y}. \quad (10)$$

In particular,

$$\tau_{2,y} = \rho_{2,y} \quad \text{and} \quad \tau_{0,y} = \widetilde{\rho_{0,y}}. \quad (11)$$

4.1. **The main theorem on Tutte characters.** Let

$$P_{x,y}(G;k) = \rho^k_{x,y}(G)$$

be the image of G under the CHA morphism $(\mathcal{G}, \rho_{x,y}) \to \mathbb{F}(x,y)[k]$ of Proposition 2.2. Thus $P_{x,y}(G;k) \in \mathbb{C}(x,y)[k]$. The main theorem of this section is that $P_{x,y}(G;k)$ is itself essentially an evaluation of the Tutte polynomial.

Theorem 4.1. *We have*

$$P_{x,y}(G;k) = k^{c(G)}(x-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+x-1}{x-1}, y\right).$$

Proof. Since $P_{x,y}(G;k)$ is a polynomial in k, it suffices to prove that the identity holds for all positive integer values of k.

We have

$$P_{x,y}(G;k) = \rho^k_{x,y}(G) = \sum_{V_1 \uplus \cdots \uplus V_k = V(G)} \prod_{i=1}^{k} \rho_{x,y}(G|_{V_i}) \tag{12a}$$

$$= \sum_{V_1 \uplus \cdots \uplus V_k = V(G)} \prod_{i=1}^{k} \sum_{A_i \subseteq E(G|_{V_i})} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{12b}$$

$$= \sum_{f:V\to[k]} \prod_{i=1}^{k} \sum_{A_i \subseteq f^{-1}(i)} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{12c}$$

$$= \sum_{f:V\to[k]} \sum_{A \subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)} \tag{12d}$$

where $M(f,G)$ denotes the set of f-monochromatic edges, that is, edges $e = uv \in E(G)$ such that $f(u) = f(v)$ (including, in particular, all loops). Here the sum is over all ordered partitions of $V(G)$ into pairwise disjoint subsets (possibly empty). In order to find a recipe for $P_{x,y}(G;k)$ as a Tutte specialization, we need to know its value on edgeless graphs, and how it behaves with respect to deleting a loop, deleting a cut-edge, or deletion and contraction of an "ordinary" edge.

Step 1: Edgeless graphs. If $E(G) = \emptyset$, then $R_H(x,y) = 1$ for every subgraph $H \subseteq G$. Therefore, every summand in (12a) is 1, so $P_{x,y}(G;k)$ is just the number of ordered partitions with $n = |V(G)|$ parts, that is:

$$P_{x,y}(\overline{K_n};k) = k^n. \tag{13}$$

Step 2: Loops. Suppose G has a loop ℓ. For every ordered partition $V(G) = V_1 \uplus \cdots \uplus V_k$, let V_i be the part that contains the endpoint of ℓ. Then $\rho_{x,y}(G|_{V_i}) = y\rho_{x,y}((G-\ell)|_{V_i})$, and we conclude that

$$P_{x,y}(G;k) = y \cdot P_{x,y}(G-\ell;k). \tag{14}$$

Step 3: Nonloop edges. Suppose G has a nonloop edge e (possibly a cut-edge) with endpoints u,v. For a function $f : V \to [k]$, if $f(u) \neq f(v)$ then $M(f,G-e) = M(f,G)$, while if $f(u) = f(v)$ then $M(f,G-e) = M(f,G) \setminus \{e\}$. For every edge set $A \subseteq M(f,G)$ containing e, the edge set $B = A \setminus \{e\} \subseteq M(f,G/e)$ satisfies $\mathrm{null}(B) = \mathrm{null}(A)$ and $\mathrm{rk}(B) = \mathrm{rk}(A) - 1$;

moreover, the correspondence between A and B is a bijection. Therefore,

$$\begin{aligned}
P_{x,y}(G;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)},\\
P_{x,y}(G-e;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)},\\
P_{x,y}(G;k)-P_{x,y}(G-e;k) &= \sum_{\substack{f:V\to[k]:\\ e\in M(f,G)}} \sum_{\substack{A\subseteq M(f,G):\\ e\in A}} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}\\
&= \sum_{\substack{f:V\to[k]:\\ f(u)=f(v)}} \sum_{B\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(B)+1}(y-1)^{\mathrm{null}(B)}\\
&= (x-1)P_{x,y}(G/e;k).
\end{aligned}$$

To put this recurrence in a more familiar form,

$$P_{x,y}(G;k) = P_{x,y}(G-e;k) + (x-1)P_{x,y}(G/e;k). \tag{15}$$

Step 4: Cut-edges. Now suppose that $e=uv$ is a cut-edge. We have

$$P_{x,y}(G-e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}$$

and

$$P_{x,y}(G/e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}.$$

Let H be the connected component of $G-e$ containing u, and let $H'=G-H$. Then $E(G-e)=E(H)\cup E(H')$. Let the cyclic group $\mathbb{Z}_k$ act on colorings f by cycling the colors of vertices in H modulo k and fixing the colors of vertices in H'; i.e., if we fix a generator γ of $\mathbb{Z}_k$, then $(\gamma^j f)(w)=f(w)+j \pmod{k}$ for $w\in V(H)$, while $(\gamma f)(w)=f(w)$ for $w\in V(H')$. Then the set $M(f,G-e)$ is invariant under the action of $\mathbb{Z}_k$; moreover, each orbit has size k and has exactly one coloring for which $f(u)=f(v)$. In that case, contracting the edge e does not change the nullity or rank. Therefore, $P_{x,y}(G/e;k)=k^{-1}P_{x,y}(G-e;k)$, which when combined with (15) yields

$$\begin{aligned}
P_{x,y}(G;k) &= P_{x,y}(G-e;k)+(x-1)P_{x,y}(G-e;k)/k\\
&= \left(\frac{k+x-1}{k}\right)P_{x,y}(G-e;k). \qquad (16)
\end{aligned}$$

Now combining (13), (14), (15), and (16) with the "recipe theorem" [Bol98, p. 340] (replacing Bollobás' $x, y, \alpha, \sigma, \tau$ with $(k+x-1)/k$, y, k, 1, $x-1$ respectively) gives the desired result. □

4.2. **Applications to Tutte polynomial evaluations.** Theorem 4.1 has many enumerative consequences, some familiar and some less so. Many of the formulas we obtain resemble those in the work of Ardila [Ard07]; the precise connections remain to be investigated.

First, observe that setting $x = y = t$ in Theorem 4.1 yields

$$\begin{aligned} \rho^k_{t,t}(G) &= P_{t,t}(G;k) = k^{c(G)}(t-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+t-1}{t-1}, t\right) \\ &= k^{c(G)}\overline{\chi}_G(k;t) \end{aligned} \tag{17}$$

where $\overline{\chi}$ denotes Crapo's coboundary polynomial; see [MR05, p. 236] and [BO92, §6.3.F]. (As a note, the bar in the notation $\overline{\chi}$ has no relation to the bar involution on $\mathbb{X}(\mathcal{G})$.)

Corollary 4.2. *For $k \in \mathbb{Z}$ and y arbitrary, the Tutte characters $\tau_{2,y}$ and $\tau_{0,y}$ satisfy the identities*

$$(\tau_{2,y})^k(G) = k^{c(G)} T_G(k+1, y), \tag{18}$$

$$(\widetilde{\tau_{0,y}})^k(G) = k^{c(G)}(-1)^{\mathrm{rk}(G)} T_G(1-k, y). \tag{19}$$

In particular, $(\widetilde{\tau_{0,y}})^{-1} = \overline{\tau_{2,y}}$.

Proof. Setting $x = 2$ or $x = 0$ in Theorem 4.1 and applying (11), we obtain respectively

$$\begin{aligned} (\tau_{2,y})^k(G) &= (\rho_{2,y})^k(G) = P_{2,y}(G;k) = k^{c(G)} T_G(k+1, y), \\ (\widetilde{\tau_{0,y}})^k(G) &= (\rho_{0,y})^k(G) = P_{0,y}(G;k) = k^{c(G)}(-1)^{\mathrm{rk}(G)} T_G(1-k, y). \end{aligned}$$

establishing (18) and (19). In particular, setting $k = -1$ in (19) gives

$$(\widetilde{\tau_{0,y}})^{-1}(G) = (-1)^{c(G)}(-1)^{\mathrm{rk}(G)} T_G(2, y) = (-1)^{n(G)}\tau_{2,y}(G) = \overline{\tau_{2,y}}(G). \quad \square$$

Similarly, we can find combinatorial interpretations of convolution powers of the characters $\tau_{2,2}$, $\tau_{2,0}$, $\widetilde{\tau_{0,2}}$, and $\widetilde{\tau_{0,0}}$. In the last case, we recover the standard formula for the chromatic polynomial as a specialization of the Tutte polynomial (note that $\widetilde{\tau_{0,0}} = \tau_{0,0} = \zeta$).

One can deduce combinatorial interpretations of other evaluations of the Tutte polynomial. If G is connected, then substituting $y = 2$ and $k = 2$ into (18) yields $(\tau_{2,2})^2(G) = 2T_G(3, 2)$, or

$$T(G;3,2) = \frac{(\tau_{2,2} * \tau_{2,2})(G)}{2} = \sum_{U \subseteq V(G)} 2^{e(G|_U) + e(G|_{\overline{U}}) - 1}. \tag{20}$$

That is, $2T(G;3,2)$ counts the pairs (f, A), where f is a 2-coloring of G and A is a set of f-monochromatic edges.

In order to interpret more general powers of Tutte characters, we use (10) to expand the convolution power $\rho^k_{x,y}(G)$ in Theorem 4.1:

$$\begin{aligned} k^{c(G)}(x-1)^{\mathrm{rk}(G)}T_G\left(\frac{k+x-1}{x-1},y\right) &= \rho^k_{x,y}(G) \\ &= \sum_{V_1\uplus\cdots\uplus V_k=V(G)} \prod_{i=1}^k \rho_{x,y}(G_i) \\ &= \sum_{V_1\uplus\cdots\uplus V_k=V(G)} \prod_{i=1}^k (x-1)^{\mathrm{rk}(G_i)}\tau_{x/(x-1),y}(G_i) \end{aligned}$$

where $G_i = G|_{V_i}$. Note that in the special case $G = K_n$, we have $G_i \cong K_{|V_i|}$ and $\mathrm{rk}(G_i) = |V_i| - 1$ for all i, so the equation simplifies to

$$k(x-1)^{n-1}T_{K_n}\left(\frac{k+x-1}{x-1},y\right) = (x-1)^{n-k}\tau_{x/(x-1),y}(K_n)$$

or

$$(x-1)^{k-1}T_{K_n}\left(\frac{k+x-1}{x-1},y\right) = k^{-1}(\tau_{x/(x-1),y})^k(K_n). \tag{21}$$

This equation has further enumerative consequences: setting $x = 2$ gives

$$T_{K_n}(k+1,y) = \frac{1}{k}\sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1!a_2!\dots a_k!}\tau_{2,y}(K_{a_1})\dots\tau_{2,y}(K_{a_k}). \tag{22}$$

Setting $y = 0$ in (22) and observing that $\tau_{2,0}(K_a) = a!$ (the number of acyclic orientations of K_a), we get $T_{K_n}(k+1,0) = (n+k-1)!/k!$. This is not a new formula; it follows from the standard specialization of the Tutte polynomial to the chromatic polynomial [BO92, Prop. 6.3.1], together with the well-known formula $k(k-1)\cdots(k-n+1)$ for the chromatic polynomial of K_n. On the other hand, setting $y = 2$ in (22), and recalling that $\tau_{2,2}(K_a) = 2^{|E(K_a)|} = 2^{\binom{a}{2}}$, gives

$$T_{K_n}(k+1,2) \;=\; k^{-1}\sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1!a_2!\dots a_k!}2^{\binom{a_1}{2}+\cdots+\binom{a_k}{2}}. \tag{23}$$

This formula may be obtainable from the generating function for the Crapo coboundary polynomials of complete graphs, as computed by Ardila [Ard07, Thm. 4.1]; see also sequence A143543 in [Slo10]. Notice that setting $k = 2$ in (23) recovers (20).

It is natural to ask what happens when we set $x = 1$, since this specialization of the Tutte polynomial has well-known combinatorial interpretations in terms of, e.g., the chip-firing game [ML97] and parking functions [GS96]. The equations (8) and (9) degenerate upon direct substitution, but we can instead take the limit of both sides of Theorem 4.1 as $x \to 1$, obtaining (after some calculation, which we omit)

$$\rho^k_{1,y}(G) = k^{n(G)}.$$

What can be said about Tutte characters in light of Proposition 2.2? Replacing x with $(k+x-1)/(x-1)$ in Theorem 4.1, we get

$$\begin{aligned} P_{(k+x-1)/(x-1),y}(G;k) &= k^{c(G)}(k/(x-1))^{\mathrm{rk}(G)}T(G;x,y) \\ &= k^{n(G)}(x-1)^{-\mathrm{rk}(G)}T(G;x,y). \end{aligned} \tag{24}$$

One consequence is a formula for the Tutte polynomial in terms of P:

$$T(G;x,y) = k^{-n(G)}(x-1)^{\mathrm{rk}(G)}P_{(k+x-1)/(x-1),y}(G;k). \tag{25}$$

In addition, the left-hand-side of (24) — which is an element of $\mathbb{F}(x,y)[k]$ — is actually just $k^{n(G)}$ times a rational function in x and y. Setting $k = x-1$ or $k = 1-x$, we can write down simpler formulas for the Tutte polynomial in terms of P:

$$\begin{aligned} T(G;x,y) &= (x-1)^{-c(G)}P_{2,y}(G;x-1), \\ T(G;x,y) &= (-1)^{n(G)}(x-1)^{c(G)}P_{0,y}(G;1-x). \end{aligned}$$

5. A reciprocity relation between K_n and K_m

For each nonzero scalar $c \in \mathbb{F}$, there is a character ξ_c on $\mathcal{G}$ defined by $\xi_c(G) = c^{n(G)}$. In this concluding section, we list some basic identities involving convolution powers of these characters and their interactions with the character ζ. The main result, Proposition 5.1, is a "reciprocity" relation between the complete graphs K_n and K_m.

First, let $c, d \in \mathbb{F}$ be arbitrary nonzero scalars, and let k be an integer. The definition of convolution, together with a straightforward application of the binomial theorem, yields the identities

$$\xi_c * \xi_d = \xi_{c+d}, \qquad \xi_c^k = \xi_{ck}, \qquad \xi_c^{-1} = \xi_{-c} = \overline{\xi_c}.$$

In particular, the characters ξ_c form a subgroup of $\mathbb{X}(\mathcal{G})$ isomorphic to the additive group $\mathbb{F}$. Another easily obtained fact is the following: for every graph G,

$$(\zeta * \xi_c)(G) = \sum_{\text{cocliques } Q} c^{|V(G)|-|Q|}.$$

Proposition 5.1. *For all $n, m \in \mathbb{Z}_{\geq 0}$, we have*

$$(\zeta^n * \xi_1)(K_m) = (\zeta^m * \xi_1)(K_n).$$

Proof. Consider the action of the character $\zeta^n * \xi_1$ on the graph K_m:

$$\begin{aligned} (\zeta^n * \xi_1)(K_m) &= \sum_{V \subseteq [m]} \zeta^n(K_m|_V)\xi_1(K_m - V) \\ &= \sum_{j \in \mathbb{Z}} \binom{m}{j} \zeta^n(K_j)\xi_1(K_{m-j}) \\ &= \sum_{j \in \mathbb{Z}} \binom{m}{j}\binom{n}{j} j!. \end{aligned} \tag{26}$$

Here the summand j corresponds to $m - |V|$; note that $\zeta^n(K_j) = \binom{n}{j} j!$ is the number of n-colorings of K_j. (We are using the convention that $\binom{n}{j}$ vanishes when $j < 0$ or $j > n$.) The expression (26) is symmetric in m and n, implying the desired result. □

Corollary 5.2. *Let m, n be nonnegative integers. Then*

$$(\zeta^n * \xi_{-1})(K_m) = (-1)^{n+m}(\zeta^m * \xi_{-1})(K_n).$$

Proof. The desired identity can be obtained by applying the bar involution to both sides of Proposition 5.1 (or, equivalently, redoing the calculation, replacing ξ_1 with ξ_{-1} throughout). □

Experimental evidence indicates that

$$(\zeta^{-1} * \xi_1)(K_n) = (-1)^n D_n, \qquad (\zeta^{-1} * \xi_{-1})(K_n) = (-1)^n A_n,$$

where D_n is the number of derangements of $\{1, 2, \ldots, n\}$ and A_n is the number of arrangements (sequences A000166 and A000522 of [Slo10], respectively). More generally, we have conjectured that for every scalar c and integer k, the exponential generating function for $(\zeta^k * \xi_c)(K_n)$ is

$$\sum_{n \geq 0} (\zeta^k * \xi_c)(K_n) \frac{x^n}{n!} \;=\; e^{cx}(1+x)^k \tag{27}$$

(see [Sta97, Example 2.2.1], [Sta99, Example 5.1.2]). In fact, formula (27) follows from independent, unpublished work of Aguiar and Ardila [AA].

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