Math 824, Fall 2016
Problem Set #2

Instructions: Do all problems and typeset them in LaTeX. E-mail your final PDF file to Jeremy at `jlmartin@ku.edu` by **Friday, September 16, 11:59pm**. You are encouraged to use the **header file** available from the course website. Citations to the **lecture notes** refer to the version of **September 9**. Check that you have the most current version.

Exercise 2.1. Let $\mathcal{A}$ be a finite family of sets. For $\mathcal{A}' \subset \mathcal{A}$, define $\cup\mathcal{A}' = \bigcup_{A \in \mathcal{A}'} A$. Let $U(\mathcal{A}) = \{\cup\mathcal{A}' \mid \mathcal{A}' \subseteq \mathcal{A}\}$, considered as a poset ordered by inclusion.

(a) Prove that $U(\mathcal{A})$ is a lattice. (Hint: Don't try to specify the meet operation explicitly.)
(b) Construct a set family $\mathcal{A}$ such that $U(\mathcal{A})$ is isomorphic to weak Bruhat order on $\mathfrak{S}_3$ (see Example 2.11).
(c) Construct a set family $\mathcal{A}$ such that $U(\mathcal{A})$ is not ranked.
(d) [Optional] Is every finite lattice of this form? I do not know the answer to this, or if the answer is known. If you can resolve the question, great, but I'd also be interested in any partial results can you come up with? (E.g., for which classes of lattices can you prove it?)

Exercise 2.2. Prove that the two formulations of distributivity of a lattice L are equivalent, i.e.,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in L \qquad \iff \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x,y,z \in L.$$

Exercise 2.3. In the previous problem set, you proved that the divisor lattice D_n is distributive. Characterize the lattices D_n among all distributive lattices in terms of the corresponding posets of join-irreducibles. (In other words, prove a statement of the form "A distributive lattice L is isomorphic to a divisor lattice if and only if $\operatorname{Irr}(L)$ is ______________.")

Exercise 2.4. Let Y be Young's lattice (which we know is distributive).

(a) Describe the elements of $\operatorname{Irr}(Y)$. If $\lambda = \mu_1 \vee \cdots \vee \mu_k$ is an irredundant decomposition into join-irreducibles, then what quantity does k correspond to in the Ferrers diagram of λ?
(b) Count the maximal chains in the interval $[\emptyset, \lambda] \subset Y$ if the Ferrers diagram of λ is a $2 \times n$ rectangle.
(c) Ditto if λ is a hook shape (i.e., $\lambda = (n+1, 1, 1, \ldots, 1)$, with a total of m copies of 1).

Exercise 2.5. Fill in the details in the proof of Theorem 2.21 (the FTFDL) by showing the following facts.

(a) For a finite distributive lattice L, show that the map $\phi : L \to J(\operatorname{Irr}(L))$ given by
$$\phi(x) \;=\; \langle p \mid p \in \operatorname{Irr}(L),\ p \leq x \rangle$$
is indeed a lattice isomorphism.
(b) For a finite poset P, show that an order ideal in P is join-irreducible in $J(P)$ if and only if it is principal (i.e., generated by a single element).

Exercise 2.6. Fill in the details in the proof of Theorem 1.23 in the notes (the Fundamental Theorem of Polytopes) by proving the four assertions marked as "One must show". (If necessary, you may use Minkowski's Hyperplane Separation Theorem, which states that if $S \subseteq \mathbb{R}^n$ is a convex set and $\mathbf{y} \notin S$, then there exists a hyperplane separating S from $\mathbf{y}$ — or equivalently a linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(\mathbf{y}) > 0$ and $\ell(\mathbf{x}) < 0$ for all $\mathbf{x} \in S$.)

Hints: The trickier ones are the second and fourth "One must show" statements. For the second, let $\mathbf{x}$ be a point of P that is not a vertex. Induct on the dimension of the unique minimal face $F_{\mathbf{x}}$ of P containing $\mathbf{x}$, and use multiple parts of Prop. 1.26. For the fourth one, the hard part is to show that $P^{**} \subseteq P$. Don't try to show directly that every point in P^{**} is a convex combination of the vertices of P. Instead, use the Hyperplane Separation Theorem to show that if $\mathbf{x} \in P^{**} \setminus P$, then there is a vector $\mathbf{a}$ such that $\mathbf{a} \cdot \mathbf{x} > 1 > \mathbf{a} \cdot \mathbf{w}$ for all $\mathbf{w} \in P$, then derive a contradiction.