Math 824, Fall 2012
Problem Set #3

Instructions: Type up your solutions using LaTeX. There is a header file at `http://www.math.ku.edu/~jmartin/math824/header.tex` with macros that may be useful. E-mail me (`jmartin@math.ku.edu`) the PDF file under the name {*your-name*}*3.pdf*.

Deadline: **3:00 PM on Friday, October 5.**

Problem #1 Determine, with proof, all pairs of integers $k \leq n$ such that there exists a graph G with $M(G) \cong U_k(n)$. (Recall that $U_k(n)$ is the matroid on $E = [n]$ such that every subset of E of cardinality k is a basis.)

Problem #2 Let X and Y be disjoint sets of vertices, and let B be an X, Y-bipartite graph: that is, every edge of B has one endpoint in each of X and Y. For $V = \{x_1, \dots, x_n\} \subset X$, a *transversal* of V is a set $W = \{y_1, \dots, y_n\} \subset Y$ such that $x_i y_i$ is an edge of B. (The set of all edges $x_i y_i$ is called a *matching*.) Let $\mathscr{I}$ be the family of all subsets of X that have a transversal; it is immediate that $\mathscr{I}$ is a simplicial complex.

Prove that $\mathscr{I}$ is in fact a matroid independence system by verifying that the donation condition holds. (Suggestion: Write down an example or two of a pair of independent sets I, J with $|I| < |J|$, and use the corresponding matchings to find a systematic way of choosing a vertex that J can donate to I.) These matroids are called *transversal matroids*; along with linear and graphic matroids, they are the other "classical" examples of matroids in combinatorics.)

Problem #3 Let $G = (V, E)$ be a graph with n vertices and c components. For a vertex coloring $f : V \to \mathbb{N}$, let $i(f)$ denote the number of "improper" edges, i.e., whose endpoints are assigned the same color. *Crapo's coboundary polynomial* of G is

$$\bar{\chi}_G(q;t) \;=\; q^{-c} \sum_{f:V\to[q]} t^{i(f)}.$$

This is evidently a stronger invariant than the chromatic polynomial of G, which can be obtained as $q\bar{\chi}_G(q,0)$. In fact, the coboundary polynomial provides the same information as the Tutte polynomial.

Prove that

$$\bar{\chi}_G(q;t) \;=\; (t-1)^{n-c} T_G\left(\frac{q+t-1}{t-1}, t\right)$$

by finding a deletion/contraction recurrence for the coboundary polynomial.

Problem #4 Let P be a chain-finite poset. The *kappa function* of P is the element of the incidence algebra $I(P)$ defined by $\kappa(x,y) = 1$ if $x \lessdot y$, $\kappa(x,y) = 0$ otherwise.

(#4a) Give a condition on κ that is equivalent to P being ranked.

(#4b) Give combinatorial interpretations of $\kappa * \zeta$ and $\zeta * \kappa$.

(See next page for Problem #5.)

Problem #5 Let Π_n be the lattice of set partitions of $[n]$. Recall that the order relation on Π_n is given as follows: if $\pi, \sigma \in \Pi_n$, then $\pi \leq \sigma$ if every block of π is contained in some block of σ (for short, "π refines σ"). In this problem, you're going to calculate the number $\mu_n := \mu_{\Pi_n}(\hat{0}, \hat{1})$.

(#5a) Calculate μ_n by brute force for $n = 1, 2, 3, 4$. Make a conjecture about the value of μ_n in general.

(#5b) Define a function $f : \Pi_n \to \mathbb{Q}[x]$ as follows: if X is a finite set of cardinality x, then

$$f(\pi) = \#\big\{h : [n] \to X \quad \big| \quad h(s) = h(s') \iff s, s' \text{ belong to the same block of } \pi\big\}.$$

For example, if $\pi = \hat{1} = \{\{1, 2, \ldots, n\}\}$ is the one-block partition, then $f(\pi)$ counts the constant functions from $[n]$ to X, so $f(\pi) = x$. Find a formula for $f(\pi)$ in general.

(#5c) Let $g(\pi) = \sum_{\sigma \geq \pi} f(\sigma)$. Prove that $g(\pi) = x^{|\pi|}$ for all $\pi \in \Pi_n$. (Hint: What kinds of functions are counted by the sum?)

(#5d) Apply Möbius inversion and an appropriate substitution for x to calculate μ_n.