Math 824, Fall 2012
Problem Set #2 (minor revisions 9/17/12)

Instructions: Type up your solutions using LaTeX. There is a header file at `http://www.math.ku.edu/~jmartin/math824/header.tex` with macros that may be useful. E-mail me (`jmartin@math.ku.edu`) the PDF file under the name {*your-name*}*2.pdf*.

Deadline: **3:00 PM on Friday, September 21.**

Problem #1 Prove that the partition lattice Π_n is a geometric lattice. (Hint: Construct a collection of vectors $S = \{s_{ij} \mid 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$ such that $L(S) \cong \Pi_n$.)

Problem #2 Let $\mathbb{F}$ be a field, let $n \in \mathbb{N}$, and let S be a finite subset of the vector space $\mathbb{F}^n$. Recall the definitions of the lattices $L(S)$ and $L^{\text{aff}}(S)$ (lecture notes,[1] p.17). For $s = (s_1, \dots, s_n) \in S$, let $\hat{s} = (1, s_1, \dots, s_n) \in \mathbb{F}^{n+1}$, and let $\hat{S} = \{\hat{s} \mid s \in S\}$. Prove that $L(\hat{S}) = L^{\text{aff}}(S)$. (This is useful because it saves us a dimension — e.g., many geometric lattices of rank 3 can be represented conveniently as affine point configurations in $\mathbb{R}^2$.)

Problem #3 Prove the equivalence of the two forms of the basis exchange condition (lecture notes, p.24). (Hint: Consider $|B \setminus B'|$.)

Problem #4 Prove Proposition 2.21 (lecture notes, p.25) which describes the equivalence between matroid independence systems and matroid circuit systems.

Problem #5 Let M be a matroid on ground set E with rank function $r : 2^E \to \mathbb{N}$. Let M^* be the dual matroid to M and let $r^* : 2^E \to \mathbb{N}$ be its rank function. Find a formula for r^* in terms of r. (I.e., if I know $r(A)$ for every $A \subseteq E$, how do I calculate $r(B)$ for a given $B \subseteq E$?)

Problem #6 Let E be a finite set and let Δ be an *abstract simplicial complex on* E; that is, a family of subsets with the property that if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$. Let $w : E \to \mathbb{R}_{\geq 0}$ be any function; think of $w(A)$ as the "weight" of A. For $A \subseteq E$, define $w(A) = \sum_{e \in A} w(e)$. Consider the problem of maximizing $w(A)$ over all maximal[2] elements $A \in \Delta$ (also known as *facets* of Δ). A naive approach to try to produce such a set A, which may or may not work for a given Δ and w, is the following *greedy algorithm*:

(1) Let $A = \emptyset$.
(2) If A is a facet of Δ, stop. Otherwise, find $e \in E \setminus A$ of maximal weight such that $A \cup \{e\} \in \Delta$ (if there are several such e, pick one at random), and replace A with $A \cup \{e\}$.
(3) Repeat step 2 until A is a facet of Δ.

(#6a) Construct a simplicial complex and a weight function for which this algorithm does not produce a facet of maximal weight. (Hint: The smallest example has $|E| = 3$.)

(#6b) Prove that the following two conditions are equivalent:

- The greedy algorithm produces a facet of maximal weight for *every* weight function w.
- Δ is a matroid independence system.

[1] References to the lecture notes are to the version of 9/6/12 and may be off by a page or two if I make subsequent edits.

[2] Recall that "maximal" means "not contained in any other element of Δ", which is logically weaker than "of largest possible cardinality".