Math 821, Spring 2018
Problem Set #5
Deadline: Thursday, May 3, 11:59pm

Instructions: Typeset your solutions in LaTeX. You are encouraged to use the Math 821 header file. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name and the problem set number (e.g., `Mirzakhani5.pdf`). Collaboration is encouraged, but each student must write up his or her solutions independently and acknowledge all collaborators.

Problem #1 Let $n \leqslant d \geqslant 0$ and let $X = \Delta^{n,d}$ denote the d-skeleton of the n-dimensional simplex (with $n+1$ vertices $v_0, v_1, \ldots, v_n$). *Without writing down any explicit simplicial boundary matrices*, prove that the reduced homology groups of X are given by

$$\tilde{H}_k(X) = \begin{cases} \mathbb{Z}^{\binom{n}{d+1}} & \text{if } k = d, \\ 0 & \text{if } k < d. \end{cases}$$

(You may want to first use a computer to convince yourself that the result is correct.)

Problem #2 [Hatcher p.156 #9bc] Compute the homology groups of the following spaces:

(#2a) $\mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$.

(#2b) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks, then identifying all three resulting circles together via homomorphisms preserving clockwise orientations of the circles.

Problem #3 [Hatcher p.156 #11] Let K be the 3-dimensional cell complex obtained from the cube I^3 by identifying each pair of opposite faces via a one-quarter twist. (See exercise #14 on p.54.) Compute the homology groups $\tilde{H}_n(K;\mathbb{Z})$ and $\tilde{H}_n(K;\mathbb{Z}_2)$ for $n > 0$.

Problem #4 [Hatcher p.157 #28(a), modified] (a) Use a Mayer-Vietoris sequence to compute the homology groups of the space X obtained from a torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ by attaching a Möbius band M via a homeomorphism from the boundary circle C of M to the circle $\mathbb{S}^1 \times \{x_0\}$ in the torus.

(b) How does the answer change if C is attached to a closed loop that wraps k times around the first circle (i.e., via the path $f : I \to \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C}$ given by $f(t) = (e^{2\pi i k t}, 1)$)?

Problem #5 [Hatcher p.205 #8(a)] Many basic homology arguments work equally well for cohomology, with maps in the opposite direction. In particular, for all coefficient rings R and numbers $i \leqslant n$, compute $H^i(S^n; R)$ by induction on n in two ways: using the long exact sequence of a pair, and using the Mayer-Vietoris sequence. (We didn't explicitly talk about these sequences, but they are essentially the same as the homology versions with the arrows reversed; see §3.1 of Hatcher for details if you are not confident about reversing the arrows yourself.)

Problem #6 Let X and Y be connected spaces, so that $H^0(X;R) \cong H^0(Y;R) \cong H^0(X \vee Y;R) \cong R$.

(#6a) Show that $H^k(X \vee Y;R) \cong H^k(X;R) \oplus H^k(Y;R)$ for all $k > 0$.

(#6b) Show that if $\alpha \in H^k(X;R)$ and $\beta \in H^\ell(Y;R)$ with $k, \ell > 0$, then $\alpha \smile \beta = 0$ in $H^*(X \vee Y;R)$.

(#6c) Conclude that $\mathbb{S}^1 \times \mathbb{S}^1$ is not homeomorphic to $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$, even though their homology and cohomology groups are equal in all dimensions.