Math 821 Problem Set #3
Posted: Friday 2/25/11
Due date: Monday 3/7/11

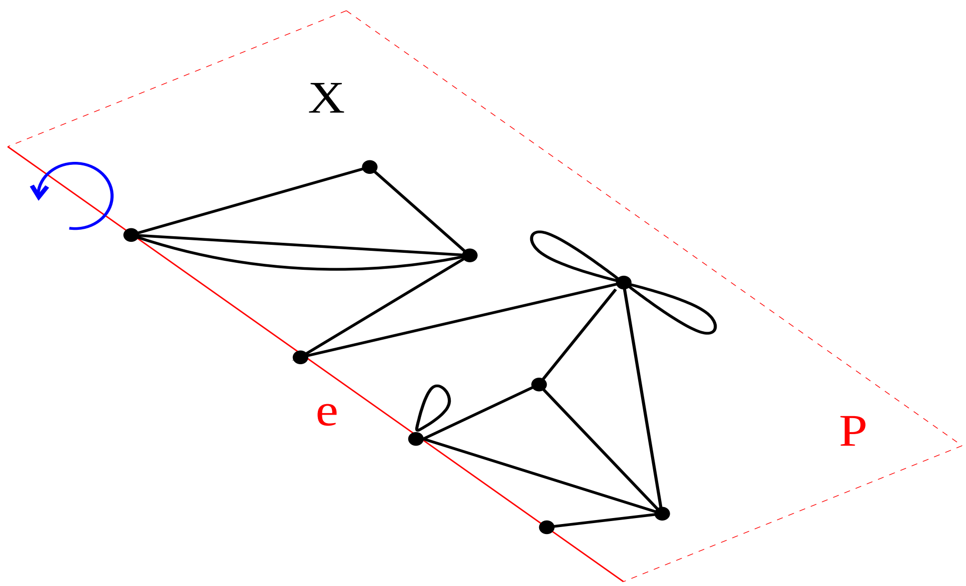
Problem #1 Let p, q be distinct points on S^2, and let X be the space obtained by gluing them together. Determine the homotopy type of X.

Problem #2 For $k \geq 1$, let T_n denote the n-holed torus.

(#2a) Construct a cell complex structure on T_n.

(#2b) Show that T_n is homotopy-equivalent to the cell complex $S^1 \times (\bigvee_{i=1}^n S^1)$ (that is, take n circles and wedge them together at a point to get a "bouquet", then take the product with another circle).

Problem #3 (Hatcher, p.20, #22) Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge e of P in a subset of its vertices. (See figure below.) Describe the homotopy type of the "surface of revolution" obtained by rotating X about e.



Problem #4 Recall that for a space X and base point $x \in X$, we have defined $\pi_1(X, x)$ to be the set of homotopy classes of x, x-paths on X — or equivalently of continuous functions $S^1 \to X$. Accordingly, define $\pi_0(X, x)$ to be the set of homotopy classes of continuous functions $S^0 \to X$.

Describe $\pi_0(X, x)$ intrinsically in terms of X.

Problem #5 How generally does the lifting property (used in the proof of $\pi_1(S^1) \cong \mathbb{Z}$) hold? That is, suppose that we have continuous functions $f : X \to Y$ and $p : Z \twoheadrightarrow Y$, and we want to find $\tilde{f}$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & Z \arrow[dd, "p"] \\
X \arrow[ur, "\tilde{f}"] \arrow[dr, "f"'] & \\
 & Y
\end{tikzcd}$$

What conditions on X, Y, Z, p guarantee that such a lift exists? (Whatever you come up with should include the case we needed for the calculation of $\pi_1(S^1)$, i.e., $X = I$, $Y = S^1$, $Z = \mathbb{R}$, $p(t) = e^{2\pi i t}$.)

Problem #6 (Hatcher, p.38, #2) Show that the change-of-basepoint homomorphism β_h (see p.28) depends only on the homotopy class of the path h.

Problem #7 (Hatcher, p.38, #7) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on *one* of the boundary circles, but not by any homotopy f_t that is stationary on *both* boundary circles.