Monday 4/28

Omega

Last time, we saw (broadly) how to use triangularity arguments to show that $\{e_\lambda\}$, $\{s_\lambda\}$, and $\{p_\lambda\}$ are bases for the ring Λ of symmetric functions (the first two $\mathbb{Z}$-bases, the second two $\mathbb{Q}$-bases). Triangularity does not work for the basis $\{h_\lambda\}$, because the complete homogeneous symmetric functions have so many terms. For example, in degree 3,

$$\begin{bmatrix} h_3 \\ h_{21} \\ h_{111} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} m_3 \\ m_{21} \\ m_{111} \end{bmatrix}$$

and it is not obvious that the base-change matrix has determinant 1 (although it does). We need a new tool to prove that $\{h_\lambda\}$ is a $\mathbb{Z}$-basis.

Define a ring endomorphism $\omega : \Lambda \to \Lambda$ by $\omega(e_i) = h_i$ for all i, so that $\omega(e_\lambda) = h_\lambda$. This is well-defined since the elementary symmetric functions are algebraically independent (recall that $\Lambda \cong R[e_1, e_2, \dots]$).

Proposition 1. *$\omega(\omega(f)) = f$ for all $f \in \Lambda$. In particular, the map ω is a ring automorphism.*

Proof. Recall the generating functions

$$E(t) = \sum_{k\geq 0} e_k t^k = \prod_{n\geq 1}(1 + tx_n), \tag{1}$$

$$H(t) = \sum_{k\geq 0} h_k t^k = \prod_{n\geq 1}(1 - tx_n)^{-1}. \tag{2}$$

Using the sum formulas in (1) and (2) gives

$$E(t)H(-t) = \sum_{n\geq 0}\sum_{k=0}^{n} e_k t^k h_{n-k}(-t)^{n-k} = \sum_{n\geq 0} t^n \sum_{k=0}^{n}(-1)^{n-k} e_k h_{n-k}. \tag{3}$$

On the other hand, the product formulas in (1) and (2) say that $E(t)H(-t) = 1$. Equating coefficients of t^n gives

$$\sum_{k=0}^{n}(-1)^{n-k} e_k h_{n-k} = 0 \qquad (\forall n \geq 1). \tag{4}$$

Applying ω, we find that

$$\begin{aligned} 0 &= \sum_{k=0}^{n}(-1)^{n-k}\omega(e_k)\omega(h_{n-k}) \\ &= \sum_{k=0}^{n}(-1)^{n-k} h_k \omega(h_{n-k}) \\ &= \sum_{k=0}^{n}(-1)^{k} h_{n-k} \omega(h_k) \\ &= (-1)^n \sum_{k=0}^{n}(-1)^{n-k} h_{n-k} \omega(h_k) \end{aligned}$$

and comparing this last expression with (4) gives $\omega(h_k) = e_k$. $\square$

Corollary 2. *$\{h_\lambda\}$ is a graded $\mathbb{Z}$-basis for Λ. Moreover, $\Lambda_R \cong R[h_1, h_2, \dots]$.*

By the way, the equation (4) can be used recursively to express the e_k's as integer polynomials in the h_k's, and vice versa.

A Bunch of Identities

The **Cauchy kernel** is the formal power series

$$\Omega = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1}.$$

As we'll see, the Cauchy kernel can be expanded in many different ways in terms of symmetric functions in the variable sets $\{x_i\}$ and $\{y_j\}$.

For a partition $\lambda \vdash n$, let m_i be the number of i's in λ, and define

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots, \qquad \varepsilon_\lambda = (-1)^{m_2+m_4+\cdots}.$$

For example, if $\lambda = (3,3,2,1,1,1)$ then $z_\lambda = 1^3 3!\, 2^1 1!\, 3^2 2! == 216$. The notation comes from the fact that this is the size of the centralizer of a permutation $\sigma \in \mathfrak{S}_n$ with cycle-shape λ (that is, the group of permutations that commute with σ). Meanwhile, ε_λ is just the sign of a permutation with cycle-shape λ.

Proposition 3. *We have the identities*

$$\prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda}, \tag{5}$$

$$\prod_{i,j\geq 1} (1 + x_i y_j) = \sum_\lambda e_\lambda(x) m_\lambda(y) = \sum_\lambda \varepsilon_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda}, \tag{6}$$

where the sums run over all partitions λ.

Proof. For the first identity in (5),

$$\begin{aligned}
\prod_{i,j\geq 1} (1 - x_i y_j)^{-1} &= \prod_{j\geq 1} \left(\prod_{i\geq 1} (1 - x_i t)^{-1} \Big|_{t=y_j} \right) \\
&= \prod_{j\geq 1} \left(\sum_{k\geq 0} h_k(x) t^k \Big|_{t=y_j} \right) \\
&= \prod_{j\geq 1} \sum_{k\geq 0} h_k(x)\, y_j^k \qquad (7) \\
&= \sum_\lambda h_k(x) m_k(y)
\end{aligned}$$

(since the coefficient on the monomial $y_1^{k_1} y_2^{k_2} \cdots$ in (7) is $h_{k_1} h_{k_2} \cdots$).

For the second identity in (5), we need some more trickery. Recall that

$$\log(1+q) = \sum_{n\geq 1} (-1)^{n+1} \frac{q^n}{n} = q - \frac{q^2}{2} + \frac{q^3}{3} - \cdots$$

Therefore,

$$\begin{aligned}
\log \prod_{i,j\geq 1}(1-x_iy_j)^{-1} &= -\log \prod_{i,j\geq 1}(1-x_iy_j) \;=\; -\sum_{i,j\geq 1}\log(1-x_iy_j) \\
&= \sum_{i,j\geq 1}\sum_{n\geq 1}\frac{x_i^n y_j^n}{n} \;=\; \sum_{n\geq 1}\frac{1}{n}\sum_{i,j\geq 1}x_i^n y_j^n \\
&= \sum_{n\geq 1}\frac{p_n(x)p_n(y)}{n}
\end{aligned}$$

and

$$\begin{aligned}
\Omega &= \exp\left(\sum_{n\geq 1}\frac{p_n(x)p_n(y)}{n}\right) \\
&= \sum_{k\geq 0}\frac{1}{k!}\left(\sum_{n\geq 1}\frac{p_n(x)p_n(y)}{n}\right)^n \\
&= \sum_{k\geq 0}\frac{1}{k!}\left[\sum_{\lambda\vdash k}\binom{k}{\lambda}\left(\frac{p_1(x)p_1(y)}{1}\right)^{m_1}\left(\frac{p_2(x)p_2(y)}{2}\right)^{m_2}\cdots\right] \\
&= \sum_{\lambda}\frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}.
\end{aligned}$$

□

The proofs of the identities in (6) are analogous, and left to the reader.

Corollary 4. *We have*

$$h_n = \sum_{\lambda\vdash n}\frac{p_\lambda}{z_\lambda}\ ; \tag{8}$$

$$e_n = \sum_{\lambda\vdash n}\varepsilon_\lambda\frac{p_\lambda}{z_\lambda}\ ; \quad \textit{and} \tag{9}$$

$$\omega(p_\lambda) = \varepsilon_\lambda p_\lambda. \tag{10}$$

Proof. For (8), we start with the identity of (5):

$$\sum_{\lambda}h_\lambda(x)m_\lambda(y) \;=\; \sum_{\lambda}\frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}.$$

Set $y_1 = t$, and $y_k = 0$ for all $k > 1$. This kills all terms on the left side for which λ has more than one part, so we get

$$\sum_{\lambda=(n)}h_n(x)t^n \;=\; \sum_{\lambda}\frac{p_\lambda(x)t^{|\lambda|}}{z_\lambda}$$

and extracting the coefficient of t^n gives (8).

Starting with (6) and doing the same thing yields (9).

As Brian pointed out, you can't obtain (10) just by applying ω to (8) and comparing with (9), as I had mistakenly claimed in class. Here is a better reason. In what follows, ω is going to act on the x_i's while leaving the y_j's alone. Using (5) and (6), we obtain

$$\sum_\lambda \frac{\boxed{p_\lambda(x)}\, p_\lambda(y)}{z_\lambda} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \omega\left(\sum_\lambda e_\lambda(x) m_\lambda(y)\right) = \omega\left(\sum_\lambda \varepsilon_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda}\right)$$
$$= \sum_\lambda \frac{\boxed{\varepsilon_\lambda \omega(p_\lambda(x))}\, p_\lambda(y)}{z_\lambda}$$

and equating coefficients of $p_\lambda(y)/z_\lambda$, as shown, yields the desired result. □

The Hall Inner Product

Definition 1. The **Hall inner product** $\langle \cdot, \cdot \rangle$ on $\Lambda_{\mathbb{Q}}$ is defined by declaring $\{h_\lambda\}$ and $\{m_\mu\}$ to be dual bases:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

- Two bases $\{u_\lambda\}$, $\{v_\lambda\}$ are dual under the Hall inner product if and only if

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_\lambda u_\lambda v_\lambda.$$

- In particular, $\left\{ \frac{p_\lambda}{\sqrt{z_\lambda}} \mid \lambda \vdash n \right\}$ is an orthonormal basis for $\Lambda_{\mathbb{R},n}$, so $\langle \cdot, \cdot \rangle$ is an inner product — that is, a nondegenerate bilinear form.

- The involution ω is an isometry, i.e., $\langle a, b \rangle = \langle \omega(a), \omega(b) \rangle$.

It sure would be nice to have an orthonormal basis for $\Lambda_{\mathbb{Z}}$. In fact, the Schur functions are such a thing. The proof of this statement requires a marvelous combinatorial tool called the **RSK correspondence** (for Robinson, Schensted and Knuth).