Monday 4/14

Let G be a finite group. All representations are finite-dimensional and over $\mathbb{C}$.

Here's the machinery we developed on Friday:

Character formulas. We proved that

$$\chi_{\rho\oplus\rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g) \tag{1}$$

$$\chi_{\rho^*}(g) = \overline{\chi_\rho(g)} \tag{2}$$

$$\chi_{\rho\otimes\rho'}(g) = \chi_\rho(g)\chi_{\rho'}(g) \tag{3}$$

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(\rho,\rho')}(g) = \overline{\chi_\rho(g)}\ \chi_{\rho'}(g). \tag{4}$$

An important missing piece is a formula for the character of $\mathrm{Hom}_G(\rho, \rho')$.

Fixed spaces. We defined the fixed space of a representation G as $V^G = \{v \in V \mid gv = h\ \forall g \in G\}$, and observed that

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^G.$$

The inner product. For $\chi, \psi \in C\ell(G)$, we defined

$$\langle \chi,\ \psi\rangle_G = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\psi(g)$$

and proved the formulas

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g), \tag{5}$$

$$\left\langle \chi_\rho,\ \chi_{\rho'}\right\rangle_G = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho, \rho'). \tag{6}$$

Schur's Lemma and the Orthogonality Relations

What happens when ρ and ρ' are irreducible representations?

Proposition 1 (Schur's Lemma). *Let G be a group, and let (ρ, V) and (ρ', V') be finite-dimensional representations of G over a field $\mathbb{F}$.*

1. *If ρ and ρ' are irreducible, then every G-equivariant $\phi : V \to V'$ is either zero or an isomorphism.*
2. *If in addition $\mathbb{F}$ is algebraically closed, then*

$$\mathrm{Hom}_G(V, V') \cong \begin{cases} \mathbb{F} & \textit{if } \rho \cong \rho' \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For (1), recall that $\ker\phi$ and $\operatorname{im}\phi$ are G-invariant subspaces. But since ρ, rho' are simple, there are not many possibilities. Either $\ker\phi = 0$ and $\operatorname{im}\phi = W$, when ϕ is an isomorphism. Otherwise, $\ker\phi = V$ or $\operatorname{im}\phi = 0$, either of which implies that $\phi = 0$.

For (2), let $\phi \in \mathrm{Hom}_G(V, V')$. If $\rho \ncong \rho'$ then $\phi = 0$ by (1) and we're done. Otherwise, we may as well assume that $V = V'$.

Since $\mathbb{F}$ is algebraically closed, ϕ has an eigenvalue λ. Then $\phi - \lambda I$ is G-equivariant and singular, hence zero by (1). So $\phi = \lambda I$. We've just shown that the only G-equivariant maps from V to itself are scalar multiplication by some λ. $\square$

Theorem 2. *Let (ρ, V) and (ρ', V') be finite-dimensional representations of G over $\mathbb{C}$.*

(i) *If ρ and ρ' are irreducible, then*

$$\langle \chi_\rho,\, \chi_{\rho'} \rangle_G \;=\; \begin{cases} 1 & \text{if } \rho \cong \rho' \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\rho_1, \dots, \rho_n$ are distinct irreducible representations and*

$$\rho \;=\; \bigoplus_{i=1}^{n} \underbrace{(\rho_i \oplus \cdot \oplus \rho_i)}_{m_i} \;=\; \bigoplus_{i=1}^{n} \rho_i^{\oplus m_i}$$

then

$$\langle \chi_\rho,\, \chi_{\rho_i} \rangle_G = m_i, \qquad \langle \chi_\rho,\, \chi_\rho \rangle_G = \sum_{i=1}^{n} m_i^2.$$

In particular, $\langle \chi_\rho,\, \chi_\rho \rangle_G = 1$ if and only if ρ is irreducible.

(iii) *If $\chi_\rho = \chi_{\rho'}$ then $\rho \cong \rho'$.*

(iv) *If $\rho_1, \dots, \rho_n$ is a complete list of irreducible representations of G, then*

$$\rho_{\text{reg}} \cong \bigoplus_{i=1}^{n} \rho_i^{\oplus \dim \rho_i}$$

and consequently

$$\sum_{i=1}^{n} (\dim \rho_i)^2 = |G|.$$

(v) *The irreducible characters (i.e., characters of irreducible representations) form an orthonormal basis for $C\ell(G)$. In particular, the number of irreducible characters equals the number of conjugacy classes of G.*

Example 1. Find all the irreducible characters of $\mathfrak{S}_4$.

There are five conjugacy classes in $\mathfrak{S}_4$, corresponding to the cycle-shapes 1111, 211, 22, 31, and 4. The squares of their dimensions must add up to $|\mathfrak{S}_4| = 24$. The only list of five positive integers with that property is 1, 1, 2, 3, 3.

We start by writing down some characters that we know.

Cycle shape	1111	211	22	31	4
Size of conjugacy class	1	6	3	8	6
$\chi_1 = \chi_{\text{triv}}$	1	1	1	1	1
$\chi_2 = \chi_{\text{sign}}$	1	-1	1	1	-1
χ_{def}	4	2	0	1	0
χ_{reg}	24	0	0	0	0

Of course χ_{triv} and χ_{sign} are irreducible (since they are 1-dimensional). On the other hand, χ_{def} can't be irreducible because $\mathfrak{S}_4$ doesn't have a 4-dimensional irrep. Indeed,

$$\langle \chi_{\text{def}},\, \chi_{\text{def}} \rangle_G = 2$$

which means that ρ_{def} must be a direct sum of two distinct irreps. (If it were the direct sum of two copies of the unique 2-dimensional irrep, then $\langle \chi_{\text{def}},\, \chi_{\text{def}} \rangle_G$ would be 4, not 2, by (ii) of Theorem 2.) We calculate

$$\langle \chi_{\text{def}},\, \chi_{\text{triv}} \rangle_G = 1, \qquad \langle \chi_{\text{def}},\, \chi_{\text{sign}} \rangle_G = 0.$$

Therefore $\chi_3 = \chi_{\text{def}} - \chi_{\text{triv}}$ is an irreducible character.

The other 3-dimensional irreducible character is $\chi_4 = \chi_3 \otimes \chi_{\text{sign}}$; we can check that $\langle \chi_4,\, \chi_4 \rangle_G = 1$.

The other irreducible character χ_5 has dimension 2. We can calculate it from the regular character and the other four irreducibles, because

$$\chi_{\mathrm{reg}} \;=\; (\chi_1+\chi_2)+3(\chi_3+\chi_4)+2\chi_5$$

.

So here is the complete character table of $\mathfrak{S}_4$:

Cycle shape	1111	211	22	31	4
Size of conjugacy class	1	6	3	8	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	-1
χ_4	3	-1	-1	0	1
χ_5	2	0	2	-1	0

Now, the proof of Theorem 2.

Assertion (i) follows from part (2) of Schur's Lemma together with Proposition 6, and (ii) follows because the inner product is bilinear on direct sums. For (iii), Maschke's Lemma says that every complex representation ρ can be written as a direct sum of irreducibles. Their multiplicities determine ρ up to isomorphism, and can be recovered from χ_ρ by assertion (ii).

For (iv), recall that $\chi_{\mathrm{reg}}(1_G)=|G|$ and $\chi_{\mathrm{reg}}(g)=0$ for $g\neq 1_G$. Therefore

$$\left\langle \chi_{\mathrm{reg}},\, \rho_i \right\rangle_G = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\mathrm{reg}}(g)}\rho_i(g) = \frac{1}{|G|}|G|\rho_i(1_G) = \dim\rho_i$$

so ρ_i appears in ρ_{reg} with multiplicity equal to its dimension.

That the irreducible characters are orthonormal (hence linearly independent in $C\ell(G)$) follows from Schur's Lemma together with assertion (3). The hard part is to show that they in fact span $C\ell(G)$. We will do this next time.