Friday 4/11

Until further notice, G is still a finite group and all representations are finite-dimensional over $\mathbb{C}$.

New Characters from Old

In order to investigate characters, we need to know how standard vector space (or, in fact, G-module) functors such as $\oplus$ and $\otimes$ affect the corresponding characters.

Throughout, let (ρ, V), (ρ', V') be representations of G, with $V \cap V' = \emptyset$.

1. **Direct sum**.

To construct a basis for $V \oplus V'$, we can take the union of a basis for V and a bais for V'. Equivalently, we can write the vectors in $V \oplus V'$ as column block vectors:

$$V \oplus V' = \left\{ \begin{bmatrix} v \\ v' \end{bmatrix} \mid v \in V,\ v' \in V' \right\}.$$

Accordingly, define $(\rho \oplus \rho', V \oplus V')$ by

$$(\rho \oplus \rho')(h) = \left[\begin{array}{c|c} \rho(h) & 0 \\ \hline 0 & \rho'(h) \end{array}\right].$$

From this it is clear that

(1) $$\boxed{\chi_{\rho\oplus\rho'}(h) = \chi_\rho(h) + \chi_{\rho'}(h).}$$

2. **Duality**.

Recall that the *dual space* V^* of V consists of all $\mathbb{F}$-linear transformations $\phi : V \to \mathbb{F}$. Given a representation (ρ, V), there is a natural action of G on V^* defined by

$$(h\phi)(v) = \phi(h^{-1}v)$$

for $h \in G$, $\phi \in V^*$, $v \in V$. (You need to define it this way in order for $h\phi$ to be a homomorphism — try it.) This is called the **dual representation** (or **contragredient representation** ρ^*.

Proposition: For every $h \in G$,

(2) $$\boxed{\chi_{\rho^*}(h) = \overline{\chi_\rho(h)}.}$$

Proof. Choose a basis $\{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of h (since we are working over $\mathbb{C}$); say $hv_i = \lambda_i v_i$.

In this basis, $\rho(h) = \operatorname{diag}(\lambda_i)$ (i.e., the diagonal matrix whose entries are the λ_i), and in the dual basis, $\rho^*(h) = \operatorname{diag}(\lambda_i^{-1})$.

On the other hand, some power of $\rho(h)$ is the identity matrix, so each λ_i must be a root of unity, so its inverse is just its complex conjugate. □

3. **Tensor product**.

Recall that if $\{v_1, \ldots, v_n\}$, $\{v'_1, \ldots, v'_m\}$ are bases for V, V' respectively, then $V \otimes V'$ can be defined as the vector space with basis

$$\{v_i \otimes v'_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}.$$

In particular, $\dim V \otimes V' = (\dim V)(\dim V')$.

Accordingly, define a representation $(\rho \otimes \rho', V \otimes V')$ by

$$(\rho \otimes \rho')(h)(v \otimes v') = \rho(h)v \otimes v' + v \otimes \rho'(h)v'$$

or more concisely

$$h \cdot (v \otimes v') = (hv) \otimes v' + v \otimes (hv'),$$

extended bilinearly to all of $V \otimes V'$.

In terms of matrices, $(\rho \otimes \rho')(h)$ is represented by the block matrix

$$\begin{bmatrix} a_{11}B & a_{11}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

where $\rho(h) = [a_{ij}]_{i,j=1\ldots n}$ and $\rho'(h) = B$. In particular,

(3) $$\boxed{\chi_{\rho\otimes\rho'}(h) = \chi_\rho(h)\chi_{\rho'}(h).}$$

4. **Hom.**

Recall that $\mathrm{Hom}_G(V, V') = \mathrm{Hom}_G(\rho, \rho')$ is the vector space of all G-equivariant maps $\rho \to \rho'$.

Meanwhile, $\mathrm{Hom}_{\mathbb{C}}(V, W)$ can be made into a G-module by

(4) $$(h \cdot \phi)(v) = h(\phi(h^{-1}v)) = \rho'(h)\Big(\phi(\rho(h^{-1})(v))\Big).$$

for $h \in G$, $\phi \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, $v \in V$. (That is, h sends ϕ to the map $h \cdot \phi$ which acts on V as above.) You can then verify that this is a genuine group action.

In general, when G acts on a vector space V, the *subspace of G-invariants* is defined as

$$V^G = \{v \in V \mid hv = h\ \forall h \in G\}.$$

In our current setup, a map ϕ is G-equivariant if and only if $h \cdot \phi = \phi$ for all $h \in G$ (proof left to the reader). That is,

(5) $$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^G.$$

Moreover, $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$ as vector spaces, so

(6) $$\boxed{\chi_{\mathrm{Hom}(\rho,\rho')}(h) = \overline{\chi_\rho(h)}\ \chi_{\rho'}(h).}$$

The Inner Product

Recall that a **class function** is a function $\chi : G \to \mathbb{C}$ that is constant on conjugacy classes of G. Define an inner product on the vector space $C\ell(G)$ of class functions by

$$\langle \chi,\, \psi \rangle_G = \frac{1}{|G|} \sum_{h \in G} \overline{\chi(h)} \psi(h).$$

Proposition 1. *With this setup,*

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{h \in G} \chi_\rho(h) = \left\langle \chi_{\mathrm{triv}},\, \chi_\rho \right\rangle_G .$$

Proof. Define a linear map $\pi : V \to V$ by

$$\pi = \frac{1}{|G|} \sum_{h \in G} \rho(h).$$

In fact, $\pi(v) \in V^G$ for all $v \in V$, and if $v \in V^G$ then $\pi(v) = v$. That is, π is a projection from $V \to V^G$, and can be represented by the block matrix

$$\left[\begin{array}{c|c} I & 0 \\ \hline * & 0 \end{array}\right]$$

where the first and second column blocks (resp., row blocks) correspond to V^G and $(V^G)^\perp$ respectively. is now evident that $\dim_{\mathbb{C}} V^G = \operatorname{tr} \pi$, giving the first equality. The second equality follows because V^G is just the direct sum of all copies of the trivial representation occurring as G-invariant subspaces of V. □

Example 1. Suppose that ρ is a permutation representation. Then V^G is the space of functions that are constant on the orbits. Therefore, the formula becomes

$$\text{number of orbits} \;=\; \frac{1}{|G|} \sum_{h \in G} \text{number of fixed points of } h$$

which is Burnside's Lemma.

Proposition 2. $\left\langle \chi_\rho,\, \chi_{\rho'} \right\rangle_G = \dim_{\mathbb{C}} \operatorname{Hom}_G(\rho, \rho')$.

Proof.

$$\begin{aligned}
\left\langle \chi_\rho,\, \chi_{\rho'} \right\rangle_G &= \frac{1}{|G|} \sum_{h \in G} \overline{\chi_\rho(h)} \chi_{\rho'}(h) && \\
&= \frac{1}{|G|} \sum_{h \in G} \chi_{\operatorname{Hom}(\rho, \rho')}(h) && \text{(by (6))} \\
&= \dim_{\mathbb{C}} \operatorname{Hom}(\rho, \rho')^G && \text{(by Proposition 1)} \\
&= \dim_{\mathbb{C}} \operatorname{Hom}_G(\rho, \rho') && \text{(by (5)).}
\end{aligned}$$

□