Wednesday 4/9

Irreducibility, Indecomposability and Maschke's Theorem

Today, G is a finite group and all representations are finite-dimensional.

Definition 1. Let (ρ, V) be a representation of G. A vector subspace $W \subset V$ is G**-invariant** if $\rho(g)W \subset W$ (equivalently, if W is a G-submodule of V). V is **irreducible** (or **simple**, or colloquially an "irrep") if it has no proper G-invariant subspace.

For instance, any 1-dimensional representation is clearly irreducible.

It would be nice if every G-invariant subspace W had a *G-invariant complement*, i.e., another G-invariant subspace $W^\perp$ such that $W \cap W^\perp = 0$ and $W + W^\perp = V$. However, funny things can happen in positive characteristic.

Example 1. Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{F}^2$. Recall that the defining representation of $\mathfrak{S}_2 = \{12, 21\}$ is given by

$$\rho_{\text{def}}(12) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho_{\text{def}}(21) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and that

$$\rho_{\text{def}}(g)(e_1 + e_2) = \rho_{\text{triv}}(g)(e_1 + e_2), \qquad \rho_{\text{def}}(g)(e_1 - e_2) = \rho_{\text{sign}}(g)(e_1 - e_2).$$

Therefore, as we saw last time, the change of basis map

$$\phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^{-1}$$

is a G-equivariant isomorphism between ρ_{def} and $\rho_{\text{triv}} \oplus \rho_{\text{sign}}$ — <u>unless $\mathbb{F}$ has characteristic 2</u>. In that case, $W = \operatorname{span}\{e_1 + e_2\}$ is certainly G-invariant, but it has no G-invariant complement. D'oh!

Definition 2. The representation V is **decomposable** if there are G-invariant subspaces $W, W^\perp$ with $W \cap W^\perp = 0$ and $W + W^\perp = V$. Otherwise, V is **indecomposable**.

Clearly every representation can be written as the direct sum of indecomposables. Moreover, irreducible implies indecomposable. But the converse is not true in general, as Example 1 illustrates.

Fortunately, this kind of pathology does not happen in characteristic 0. Indeed, something stronger is true.

Theorem 1 (Maschke's Theorem)**.** *Let G be a finite group, and let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Then every representation $\rho : G \to GL(V)$ is* completely reducible*, that is, every G-invariant subspace has an invariant complement.*

Proof. If ρ is an irreducible representation, then there is nothing to prove. Otherwise, let W be a G-invariant subspace, and let

$$\pi : V \to W$$

be any projection (i.e., a surjective linear transformation, with nothing assumed about its behavior with respect to ρ).

For $v \in V$, define

$$\pi_G(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v). \tag{1}$$

Then $\pi_G(v) \in W$ because W is G-invariant. Moreover, for $h \in G$, we have

$$\begin{aligned}\pi_G(hv) &= \frac{1}{|G|}\sum_{g\in G} g\pi(g^{-1}hv)\\ &= \frac{1}{|G|}\sum_{g\in G}(hg)\pi((hg)^{-1}hv)\\ &= \frac{1}{|G|}h\sum_{g\in G} g\pi(g^{-1}v) \quad = \quad h\pi_G(v),\end{aligned}$$

that is, π_G is G-equivariant.

Now, define $W^\perp = \ker \pi_G$. Certainly $V \cong W \oplus W^\perp$ as vector spaces, and by G-equivariance, if $v \in W^\perp$ and $g \in G$, then $\pi_G(gv) = g\pi_G(v) = 0$, i.e., $gv \in W^\perp$. That is, $W^\perp$ is G-invariant. $\square$

Maschke's Theorem implies that a representation ρ is determined up to isomorphism by the multiplicity of each irreducible representation in ρ. By the way, implicit in the proof is the following useful fact:

Proposition 2. *Any G-equivariant map has a G-equivariant kernel and G-equivariant image.*

Characters

Definition 3. Let (ρ, V) be a representation of G over $\mathbb{F}$. Its *character* is the function $\chi_\rho : G \to \mathbb{F}$ given by

$$\chi_\rho(g) = \operatorname{tr}\rho(g).$$

Example 2. Some simple facts and some characters we've seen before:

(1) A one-dimensional representation is its own character.
(2) For any representation ρ, we have $\chi_\rho(1) = \dim\rho$, because $\rho(1)$ is the $n \times n$ identity matrix.
(3) The defining representation ρ_{def} of $\mathfrak{S}_n$ has character

$$\chi_{\text{def}}(\sigma) = \text{number of fixed points of } \sigma.$$

(4) The regular representation ρ_{reg} has character

$$\chi_{\text{reg}}(\sigma) = \begin{cases}|G| & \text{if } \sigma = 1_G\\ 0 & \text{otherwise.}\end{cases}$$

Example 3. Consider the two-dimensional representation ρ of the dihedral group $D_n = \langle r, s \mid r^n = s^2 = 0, srs = r^{-1}\rangle$ by rotations and reflections:

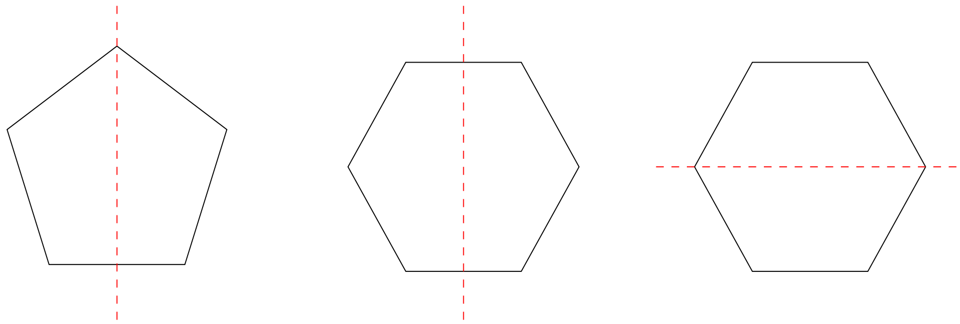
$$\rho(s) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \qquad \rho(r) = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

Its character is

$$\chi_\rho(r^i) = 2\cos i\theta \quad (0 \le i < n), \qquad \chi_\rho(sr^i) = 0 \quad (0 \le j < n).$$

On the other hand, if ρ' is the n-dimensional permutation representation on the vertices, then its character is

$$\chi_{\rho'}(g) = \begin{cases} n \text{ if } g = 1,\\ 0 \text{ if } g \text{ is a nontrivial rotation,}\\ 1 \text{ if } n \text{ is odd and } g \text{ is a reflection,}\\ 0 \text{ if } n \text{ is even and } g \text{ is a reflection through two edges,}\\ 2 \text{ if } n \text{ is even and } g \text{ is a reflection through two vertices.}\end{cases}$$



One fixed point　　No fixed points　　Two fixed points

Proposition 3. *Characters are* class functions*; that is, they are constant on conjugacy classes of G. Moreover, if $\rho \cong \rho'$, then $\chi_\rho = \chi_{\rho'}$.*

Proof. Recall from linear algebra that $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B)$ in general. Therefore,

$$\mathrm{tr}\left(\rho(hgh^{-1})\right) \;=\; \mathrm{tr}\left(\rho(h)\rho(g)\rho(h^{-1})\right) \;=\; \mathrm{tr}\left(\rho(h)\rho(g)\rho(h)^{-1}\right) \;=\; \mathrm{tr}\,\rho(g).$$

For the second assertion, let $\phi : \rho \to \rho'$ be an isomorphism, i.e., $\phi \cdot \rho(g) = \rho'(g) \cdot \phi$ for all $g \in G$ (treating ϕ as a matrix in this notation). Since ϕ is invertible, we have therefore $\phi \cdot \rho(g) \cdot \phi^{-1} = \rho'(g)$. Now take traces. □

What we'd really like is the converse of this second assertion. In fact, much, much more is true. From now on, we consider only representations over $\mathbb{C}$.

Theorem 4. *Let G be any finite group.*

(1) *If $\chi_\rho = \chi_{\rho'}$, then $\rho \cong \rho'$. That is, a representation is determined up to isomorphism by its character.*
(2) *The characters of irreducible representations form a basis for the vector space $C\ell(G)$ of all class functions of G. Moreover, this basis is orthonormal with respect to the natural Hermitian inner product defined by*

$$\langle f, f' \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g).$$

(The bar denotes complex conjugate.)
(3) *As a consequence, the number of different irreducible representations of G equals the number of conjugacy classes.*
(4) *The regular representation ρ_{reg} satisfies*

$$\rho_{\mathrm{reg}} \cong \bigoplus_{\text{irreps } \rho} \rho^{\oplus \dim \rho}$$

so in particular

$$|G| = \sum_{\text{irreps } \rho} (\dim \rho)^2.$$

Example 4. The group $G = \mathfrak{S}_3$ has three conjugacy classes, determined by cycle shapes:

$$C_1 = \{1_G\}, \qquad C_2 = \{(12),\ (13),\ (23)\}, \qquad C_3 = \{(123),\ (132)\}.$$

We'll notate a character χ by the bracketed triple $[\chi(C_1), \chi(C_2), \chi(C_3)]$.

We know two irreducible 1-dimensional characters of $\mathfrak{S}_3$, namely the trivial character $\chi_{\mathrm{triv}} = [1, 1, 1]$ and the sign character $\chi_{\mathrm{sign}} = [1, -1, 1]$.

Note that

$$\langle \chi_{\mathrm{triv}},\ \chi_{\mathrm{triv}} \rangle = 1, \qquad \langle \chi_{\mathrm{sign}},\ \chi_{\mathrm{sign}} \rangle = 1, \qquad \langle \chi_{\mathrm{triv}},\ \chi_{\mathrm{sign}} \rangle = 0.$$

Consider the defining representation. Its character is $\chi_{\mathrm{def}} = [3,1,0]$, and

$$\begin{aligned}
\langle \chi_{\mathrm{triv}},\ \chi_{\mathrm{def}} \rangle &= \frac{1}{6}\sum_{j=1}^{3} |C_j| \cdot \overline{\chi_{\mathrm{triv}}(C_j)} \cdot \chi_{\mathrm{def}}(C_j) \\
&= \frac{1}{6}\,(1\cdot 1\cdot 3 + 3\cdot 1\cdot 1 + 2\cdot 1\cdot 0) \ = \ 1, \\
\langle \chi_{\mathrm{sign}},\ \chi_{\mathrm{def}} \rangle &= \frac{1}{6}\sum_{j=1}^{3} |C_j| \cdot \overline{\chi_{\mathrm{triv}}(C_j)} \cdot \chi_{\mathrm{def}}(C_j) \\
&= \frac{1}{6}\,(1\cdot 1\cdot 3 - 3\cdot 1\cdot 1 + 2\cdot 1\cdot 0) \ = \ 0.
\end{aligned}$$

This tells us that ρ_{def} contains one copy of the trivial representation as a summand, and no copies of the sign representation. If we get rid of the trivial summand, the remaining two-dimensional representation ρ has character $\chi_\rho = \chi_{\mathrm{def}} - \chi_{\mathrm{triv}} = [2,0,-1]$.

Since

$$\langle \chi_\rho,\ \chi_\rho \rangle = \frac{1(2\cdot 2) + 3(0\cdot 0) + 2(-1\cdot -1)}{6} = 1,$$

it follows that ρ is irreducible. So, up to isomorphism, $\mathfrak{S}_3$ has two distinct one-dimensional representations $\rho_{\mathrm{triv}}, \rho_{\mathrm{sign}}$ and one two-dimensional representation ρ. Note also that

$$\chi_{\mathrm{triv}} + \chi_{\mathrm{sign}} + 2\chi_\rho = [1,1,1] + [1,-1,1] + 2[2,0,-1] = [6,0,0] = \chi_{\mathrm{reg}}.$$

New Characters from Old

In order to investigate characters, we need to know how standard vector space (or, in fact, G-module) functors such as $\oplus$ and $\otimes$ affect the corresponding characters. Throughout, let (ρ, V), (ρ', V') be representations of G, with $V \cap V' = \emptyset$.

1. Direct sum. The vectors in $V \oplus V'$ can be regarded as column block vectors $\begin{bmatrix} v \\ v' \end{bmatrix}$, for $v \in V,\ v' \in V'$. Accordingly, define $(\rho \oplus \rho', V \oplus V')$ by

$$(\rho \oplus \rho')(h) = \left[\begin{array}{c|c} \rho(h) & 0 \\ \hline 0 & \rho'(h) \end{array}\right].$$

It is clear that

(2) $$\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}.$$

Next time: Tensor product, dual, and Hom.