Wednesday 4/2

Dilworth's Theorem and Graph Theory

A *chain cover* of a poset P is a collection* of chains whose union is P.

Theorem 1 (Dilworth's Theorem)**.** *In any finite poset, the minimum size of a chain cover equals the maximum size of an antichain.*

If we switch "chain" and "antichain", the result remains true and becomes (nearly) trivial:

Proposition 2 (Trivial Proposition)**.** *In any finite poset, the minimum size of an antichain cover equals the maximum size of an chain.*

This is much easier to prove than Dilworth's Theorem.

Proof. For the $\geq$ direction, if C is a chain and $\mathcal{A}$ is an antichain cover, then no antichain in $\mathcal{A}$ can contain more than one element of C, so $|\mathcal{A}| \geq |C|$. On the other hand, let

$$A_i = \{x \in P \mid \text{ the longest chain headed by } x \text{ has length } i\};$$

then $\{A_i\}$ is an antichain cover whowe cardinality equals the length of the longest chain in P. $\square$

These theorems have graph-theoretic consequences.

The *chromatic number* $\chi(G)$ of a graph G is the smallest number k such that G has a proper k-coloring. The *clique number* $\omega(G)$ is the largest size of a clique in G (a set of pairwise adjacent vertices). Since each vertex in a clique must be assigned a different color, it follows that

$$\chi(G) \geq \omega(G). \tag{1}$$

always; however, equality need not hold (for instance, for a cycle of odd length). The graph G is called **perfect** if $\omega(H) = \chi(H)$ for every induced subgraph $H \subseteq G$.

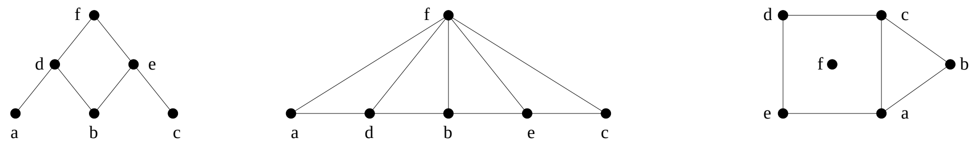
Definition 1. Let P be a finite poset. Its *comparability graph* G_P to be the graph G with vertices P and edges

$$\{xy \mid x \leq y \text{ or } x \geq y\}.$$

*It doesn't matter whether or not we require the chains to be pairwise disjoint.

Equivalently, G_P is the underlying undirected graph of the transitive closure of the Hasse diagram of P. The *incomparability graph* $\overline{G_P}$ is the complement of G_P; that is, x, y are adjacent if and only if they are incomparable.

For example, if P is the poset whose Hasse diagram is shown on the left, then G_P is P plus the edges



P $\qquad$ G_P $\qquad$ $\overline{G_P}$

A chain in P corresponds to a clique in G_P and to a coclique in $\overline{G_P}$. Likewise, an antichain in P corresponds to a coclique in G_P and to a clique in $\overline{G_P}$.

Observe that a covering of the vertex set of a graph by cocliques is exactly the same thing as a proper coloring. Therefore, the Trivial Proposition and Dilworth's Theorem say respectively that

Theorem 3. *Comparability and incomparability graphs of posets are perfect.*

Theorem 4 (Perfect Graph Theorem; Lovász 1972)**.** *Let G be a finite graph. Then G is perfect if and only if $\bar{G}$ is perfect.*

Theorem 5 (Strong Perfect Graph Theorem; Seymour/Chudnovsky 2002)**.** *Let G be a finite graph. Then G is perfect if and only if it has no "obvious bad counterexamples", i.e., induced subgraphs of the form C_r or $\bar{C}_r$, where $r \geq 5$ is odd.*

The Greene-Kleitman Theorem

There is a wonderful generalization of Dilworth's theorem due to C. Greene and D. Kleitman (1976).

Theorem 6. *Let P be a finite poset. Define two sequences of positive integers*

$$\lambda = (\lambda_1, \lambda_2, \dots), \lambda_\ell), \qquad \mu = (\mu_1, \mu_2, \dots, \mu_m)$$

by

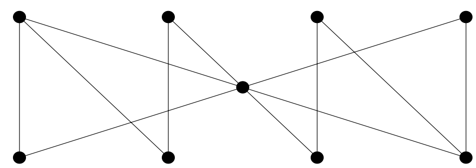
$$\lambda_1 + \cdots + \lambda_k = \max\left\{|C_1 \cup \cdots \cup C_k| : \ C_i \subseteq P \textit{ chains } \right\},$$
$$\mu_1 + \cdots + \mu_k = \max\left\{|A_1 \cup \cdots \cup A_k| : \ A_i \subseteq P \textit{ disjoint antichains } \right\}.$$

Then:

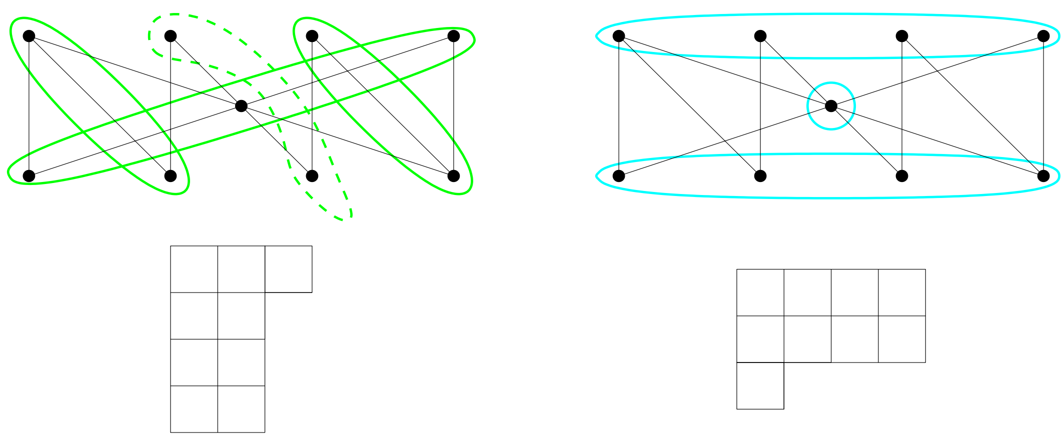
(1) λ *and* μ *are both* partitions *of* $|P|$, *i.e., weakly decreasing sequences whose sum is* $|P|$.
(2) λ *and* μ *are* conjugates, *i.e.,*

$$\mu_i = \#\{j \mid \lambda_j \geq i\}.$$

For example, consider the following poset:



Then $\lambda = (3, 2, 2, 2)$ and $\mu = (4, 4, 1)$:



Dilworth's Theorem is now just the special case $\mu_1 = \ell$.