Monday 3/31

Network Flows

Definition: A **network** is a directed graph $N = (V, E)$ with the following additional data:

- A distinguished **source** $s \in V$ and **sink** $t \in V$.
- A **capacity** function $c : E \to \mathbb{N}$.

A **flow** on N is a function $f : E \to \mathbb{N}$ that satisfies the **capacity constraints**

$$0 \leq f(e) \leq c(e) \qquad \forall e \in E \tag{1}$$

and the **conservation constraints**

$$f^-(v) = f^+(v) \qquad \forall v \in V \setminus \{s, t\} \tag{2}$$

where

$$f^-(v) = \sum_{e=\overrightarrow{uv}} f(e), \qquad\qquad f^+(v) = \sum_{e=\overrightarrow{vw}} f(e).$$

The **value** of a flow f is the net flow into the sink:

$$|f| = f^-(t) - f^+(t) = f^+(s) - f^-(s).$$

Let $S, T \subset V$ with $S \cup T = V$, $S \cap T = \emptyset$, $s \in S$, and $t \in T$. The corresponding **cut** is

$$[S, T] \;=\; \{\overrightarrow{st} \in E \mid s \in S, t \in \bar{S}\}$$

and the **capacity** of the cut is

$$c(S, T) = \sum_{e \in E} c(e).$$

We proved the main result last time:

Theorem 1 (Max-Flow/Min-Cut Theorem)**.** *Let f be a flow of maximum value and let $[S, T]$ be a cut of minimum capacity. Then $|f| = c(S, T)$.*

Acyclic and Partitionable Flows

Definition 1. A flow f is *acyclic* if, for every directed cycle $C \subset D$, i.e., every set of edges

$$C \;=\; \{\overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \ldots, \overrightarrow{x_{n-1}x_n}, \overrightarrow{x_nx_1}\},$$

there is some $e \in C$ for which $f(e) = 0$.

A flow f is *partitionable* if there is a collection of s,t-paths $P_1, \ldots, P_{|f|}$ from such that for every $e \in E$,

$$f(e) = \#\{i \mid e \in P_i\}.$$

(Here "s,t-path" means "path from s to t".)

Proposition 2.
- *For every flow, there exists an acyclic flow with the same value.*
- *Every acyclic flow is partitionable.*

Proof. Suppose that some directed cycle C has positive flow on every edge. Let $k = \min\{f(e) \mid e \in C\}$. Define $\tilde{f} : E \to \mathbb{N}$ by

$$\tilde{f}(e) = \begin{cases} f(e) - k \text{ if } e \in C, \\ f(e) \text{ if } e \notin C. \end{cases}$$

Then it is easy to check that $\tilde{f}$ is a flow, and that $|\tilde{f}| = |f|$. If we repeat this process, it must eventually stop (because the positive quantity $\sum_{e \in E} f(e)$ decreases with each iteration), which means that the resulting flow is acyclic. This proves (1).

Given an acyclic flow f, find an s,t-path P_1 along which all flow is positive. Decrement the flow on each edge of P_1; doing this will also decrement $|f|$. Now repeat this for an s,t-path P_2, etc. Eventually, we partition f into a collection of s,t-paths of cardinality $|f|$. □

Applications of the Max-Flow/Min-Cut Theorem

Let G be a graph or directed graph, and let $s,t \in V(G)$. A family of s,t-paths $\{P_1, \dots, P_n\}$ in G is *vertex-disjoint* if $V(P_i) \cap V(P_j) = \{s,t\}$ for all i,j, and is *edge-disjoint* if $E(P_i) \cap E(P_j) = \emptyset$ for all i,j. Every vertex-disjoint family is edge-disjoint, but the converse is not true.

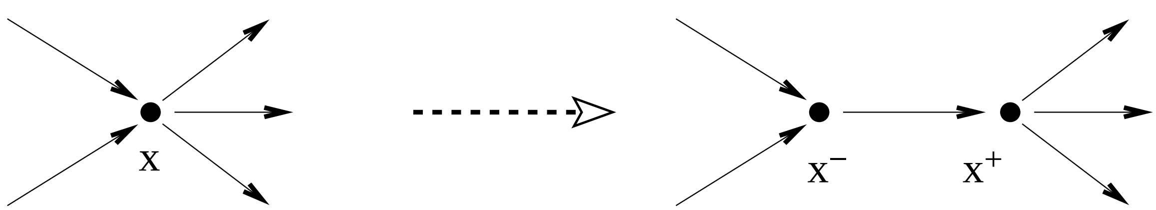
An *s,t-vertex cut* is a set $X \subseteq V(G)$ such that $G - X$ contains no s,t-path. Likewise, an *s,t-edge cut* is a set $A \subseteq E$ such that $G - A$ contains no s,t-path.

Theorem 3 (Menger's Theorem)**.** *Let G be a graph or directed graph and let $s,t \in V(G)$. Then the maximum cardinality of a vertex-disjoint (resp., edge-disjoint) family of s,t-paths equals the minimum cardinality of an s,t-vertex cut (resp., edge cut). (In the former case, we assume s,t are not adjacent.)*

Proof. First of all, an undirected graph can be considered as a digraph by replacing each edge xy with a pair of antiparallel edges $\overrightarrow{xy}, \overrightarrow{yx}$. So we may as well consider only the directed setting.

If we regard G as a network with source s and sink t, in which every edge has capacity 1, then the edge-version of Menger's Theorem is immediate from the Max-Flow/Min-Cut Theorem and Proposition 2.

For the vertex version, we need to do a little surgery on G before applying Max-Flow/Min-Cut. The trick is to separate each vertex $x \in V(G) \setminus \{s,t\}$ into an "inbox" x^- and an "out-terminal" x^+ with a bottleneck between them, so that only one path can pass through each vertex.



Specifically, define a digraph N by

$$\begin{aligned} V(N) &= \{s,t\} \cup \{x^-, x^+ \mid x \in V(G) \setminus \{s,t\}, \\ E(N) &= \{\overrightarrow{sx^-} \mid \overrightarrow{sx} \in E(G)\} \cup \{\overrightarrow{x^+t} \mid \overrightarrow{xt} \in E(G)\} \\ &\quad \cup \{\overrightarrow{x^+y^-} \mid \overrightarrow{xy} \in E(G)\} \\ &\quad \cup \{\overrightarrow{x^-x^+} \mid x \in V(G)\}, \end{aligned}$$

and regard it as a network with source s and sink t and capacity function

$$c(e) = \begin{cases} 1 & \text{if } e = \overrightarrow{x^-x^+} \text{ for some } x \in V(G), \\ \infty & \text{otherwise.} \end{cases}$$

Then an s,t-cut in N contains only finite-capacity edges, hence corresponds to an s,t-vertex cut in G. Now applying Max-Flow/Min-Cut gives the desired result. □

Back to Algebraic Combinatorics

Here is two related min-max results on posets with the same flavor as the Max-Flow/Min-Cut Theorem.

A *chain cover* of a poset P is a collection of chains whose union is P. The minimum size of a chain cover is called the *width* of P.

Theorem 4 (Dilworth's Theorem)**.** *Let P be a finite poset. Then*

$$\textit{width}(P) \;=\; \max\big\{s \mid P \textit{ has an antichain of size } s\big\}.$$

Dilworth's Theorem can be proven using Max-Flow/Min-Cut, but it involves a bit more work, so here is a poset-theoretic proof instead.

Proof. The "$\geq$" direction is clear, because if A is an antichain, then no chain can meet A more than once, so P cannot be covered by fewer than $|A|$ chains.

For the more difficult "$\leq$" direction, we induct on $n = |P|$. The result is trivial if $n = 1$ or $n = 2$.

Let Y be the set of all minimal elements of P, and let Z be the set of all maximal elements. Note that Y and Z are both antichains. First, suppose that no set other than Y and Z is an antichain of maximum size. Dualizing if necessary, we may assume Y is maximum. Let $y \in Y$ and $z \in Z$ with $y \leq z$. Then the maximum size of an antichain in $P' = P - \{y, z\}$ is $|Y| - 1$, so by induction it can be covered with $|Y| - 1$ chains, and tossing in the chain $\{y, z\}$ gives a chain cover of P of size $|Y|$.

Now, suppose that A is an antichain of maximum size that contains neither Y nor Z as a subset. Define

$$P^+ = \{x \in P \mid x \geq a \text{ for some } a \in A\},$$
$$P^- = \{x \in P \mid x \leq a \text{ for some } a \in A\}.$$

Then

- P^+, $P^- \neq \emptyset$ (otherwise A equals Z or Y).
- $P^+ \cup P^- = P$ (otherwise A is contained in some larger antichain).
- $P^+ \cap P^- = A$ (otherwise A isn't an antichain).

So P^+ and P^- are posets smaller than P, each of which has A as a maximum antichain. By induction, each has a chain cover of size $|A|$. So for each $a \in A$, there is a chain $C_a^+ \subset P^+$ and a chain $C_a^- \subset P^-$ with $a \in C_a^+ \cap C_a^-$, and

$$\big\{C_a^+ \cap C_a^- \mid a \in A\big\}$$

is a chain cover of P of size $|A|$. □