Wednesday 2/27

Möbius Inversion

Let P be a poset. Recall that we have defined the *Möbius function* of P, $\mu : P \times P \to \mathbb{Z}$, by

(1) $\mu_P(x,x) = 1$ for all $x \in P$.
(2) If $x \not\leq y$, then $\mu_P(x,y) = 0$.
(3) If $x < y$, then $\mu_P(x,y) = -\sum_{z \in [x,y)} \mu_P(x,z)$.

We saw last time that if P is a product of n chains (a distributive lattice), then

$$\mu_P(\hat{0},x) = \begin{cases} (-1)^a & \text{if } x \text{ is the join of } a \text{ atoms,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mu_{\mathscr{B}_n}(\hat{0},\hat{1}) = (-1)^n$.

Also, if $L = L_n(q)$ is the (modular) subspace lattice and $f(n,q) = \mu_L(\hat{0},\hat{1})$, then we saw that $f(n,q) = (-1)^n q^{\binom{n}{2}}$ for $n \leq 4$.

Why is the Möbius function useful?

- It is the inverse of ζ in the incidence algebra (check this!)
- It implies a more general version of inclusion-exclusion called *Möbius inversion.*
- It behaves nicely under poset operations such as product.
- It has geometric and topological applications. Even just the single number $\mu_P(\hat{0},\hat{1})$ tells you a lot about a bounded poset P; it is analogous to the Euler characteristic of a topological space.

Theorem 1 (Möbius inversion formula). *Let P be a poset in which every principal order ideal is finite, and let $f,g : P \to \mathbb{C}$. Then*

$$g(x) = \sum_{y \leq x} f(y) \quad \forall x \in P \iff f(x) = \sum_{y \leq x} \mu(y,x) g(y) \quad \forall x \in P, \tag{1a}$$

$$g(x) = \sum_{y \geq x} f(y) \quad \forall x \in P \iff f(x) = \sum_{y \geq x} \mu(x,y) g(y) \quad \forall x \in P. \tag{1b}$$

Proof. "A trivial observation in linear algebra" —Stanley.

We regard the incidence algebra as acting $\mathbb{C}$-linearly on the vector space V of functions $f : P \to \mathbb{Z}$ by

$$(f \cdot \alpha)(x) = \sum_{y \leq x} \alpha(y,x) f(y),$$
$$(\alpha \cdot f)(x) = \sum_{y \geq x} \alpha(x,y) f(y).$$

for $\alpha \in I(P)$. In terms of these actions, formulas (1a) and (1b) are respectively just the "trivial" observations

$$g = f \cdot \zeta \iff f = g \cdot \mu, \tag{2a}$$
$$g = \zeta \cdot f \iff f = \mu \cdot g. \tag{2b}$$

We just have to prove that these actions are indeed actions, i.e.,

$$[\alpha * \beta] \cdot f = \alpha \cdot [\beta \cdot f] \quad \text{and} \quad f \cdot [\alpha * \beta] = [f \cdot \alpha] \cdot \beta.$$

Indeed,

$$\begin{aligned}
(f \cdot [\alpha * \beta])(y) &= \sum_{x \leq y} (\alpha * \beta)(x, y) f(x) \\
&= \sum_{x \leq y} \sum_{z \in [x,y]} \alpha(x, z)\beta(z, y) f(x) \\
&= \sum_{z \leq y} \left(\sum_{x \leq z} \alpha(x, z) f(x) \right) \beta(z, y) \\
&= \sum_{z \leq y} (f \cdot \alpha)(z)\beta(z, y) \quad = \quad ((f \cdot \alpha) \cdot \beta)(y).
\end{aligned}$$

and the other verification is analogous. □

In the case of $\mathscr{B}_n$, the proposition says that

$$g(x) = \sum_{B \subseteq A} f(B) \quad \forall A \subseteq [n] \qquad \iff \qquad f(x) = \sum_{B \subseteq A} (-1)^{|B \setminus A|} g(B) \qquad \forall A \subseteq [n]$$

which is just the inclusion-exclusion formula. So Möbius inversion can be thought of as a generalized form of inclusion-exclusion that applies to every poset.

Example 1. Here's an oldie-but-goodie: counting *derangements*, or permutations $\sigma \in \mathfrak{S}_n$ with no fixed points.

For $S \subset [n]$, let

$$\begin{aligned}
f(S) &= \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i \text{ iff } i \in S\}, \\
g(S) &= \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i \text{ if } i \in S\}.
\end{aligned}$$

It's easy to count $g(S)$ directly. If $s = |S|$, then a permutation fixing the elements of S is equivalent to a permutation on $[n] \setminus S$, so $g(S) = (n - s)!$.

It's hard to count $f(S)$ directly. However,

$$g(S) = \sum_{R \supseteq S} f(R).$$

Rewritten in the incidence algebra $I(\mathscr{B}_n)$, this is just $g = \zeta \cdot f$. Thus $f = \mu \cdot g$, or in terms of the Möbius inversion formula (1b),

$$f(S) \;=\; \sum_{R \supseteq S} \mu(S, R) g(R) \;=\; \sum_{R \supseteq S} (-1)^{|R| - |S|} (n - |R|)! \;=\; \sum_{r=s}^{n} \binom{n}{r} (-1)^{r-s} (n - r)!$$

The number of derangements is then $f(\emptyset)$, which is given by the well-known formula

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{r} (n - r)!$$

Example 2. You can also use Möbius inversion to compute the Möbius function itself. In this example, we'll do this for the lattice $L_n(q)$. As a homework problem, you can use a similar method to compute the M obius function of the partition lattice.

Let $V = \mathbb{F}_q^n$, let $L = L_n(q)$, and let X be a $\mathbb{F}_q$-vector space of *cardinality* x (yes, cardinality, not dimension!) Define

$$g(W) \quad = \quad \text{number of } \mathbb{F}_q\text{-linear maps } \phi : V \to X \text{ such that } \ker\phi \supset W \quad = \quad x^{n-\dim W}.$$

[Choose a basis B for W and extend it to a basis B' for V. Then ϕ must send every element of B to zero, but can send each of the $n - \dim W$ elements of $M' \setminus B$ to any of the x elements of X.] Let

$$f(W) \quad = \quad \text{number of } \mathbb{F}_q\text{-linear maps } \phi : V \to X \text{ such that } \ker\phi = W.$$

Then $g(W) = \sum_{U \supset W} f(U)$, so by Möbius inversion

$$f(W) = \sum_{U:\ V \supseteq U \supseteq W} \mu_L(W, U) x^{n-\dim U}.$$

In particular, if we take W to be the zero subspace $0 = \hat{0}$, we obtain

$$f(\hat{0}) \quad = \quad \sum_{U \subseteq V} \mu_L(\hat{0}, U) x^{n-\dim U}$$

$$= \quad \sum_{U \in L} \mu_L(\hat{0}, U) x^{n-r(U)} \qquad \text{(where } r = \text{rank function of } L\text{)} \tag{3a}$$

$$= \quad \#\{\text{one-to-one linear maps } \phi : V \to X\}$$

$$= \quad (x-1)(x-q)(x-q^2)\cdots(x-q^{n-1}). \tag{3b}$$

[Choose an ordered basis $\{v_1, \ldots, v_n\}$ for V, and send each v_i to a vector in X not in the linear span of $\{\phi(v_1), \ldots, \phi(v_{i-1})\}$.]

This is just an identity of polynomials (in the ring $\mathbb{Q}[x]$, if you like). So we can equate the constant coefficients in (3a) and (3b), which gives

$$\mu_{L_n(q)}(\hat{0}, \hat{1}) = (-1)^n q^{\binom{n}{2}}.$$

The Characteristic Polynomial

Definition 1. Let P be a finite graded poset with rank function r, and suppose that $r(\hat{1}) = n$. The *characteristic polynomial* of P is defined as

$$\chi(P;x) \;=\; \sum_{z \in P} \mu(\hat{0}, z) x^{n-r(z)}.$$

This is an important invariant of a poset, particularly if it is a lattice.

- We have just seen that
$$\chi(L_n(q);x) \;=\; (x-1)(x-q)(x-q^2)\cdots(x-q^{n-1}).$$
- If P is a product of n chains, then the elements
$$\chi(P;x) \;=\; \sum_{k=0}^{n}(-1)^k \binom{n}{k} \;=\; (x-1)^n.$$
- Π_n has a nice characteristic polynomial, which you will see soon.

The characteristic polynomial is a specialization of the Tutte polynomial:

Theorem 2. *Let L be a geometric lattice with atoms E. Let M be the corresponding matroid on E, and r its rank function. Then*

$$\chi(L;x) \;=\; (-1)^{r(M)} T(M;1-x,0).$$

Proof. We have

$$\begin{aligned}
(-1)^{r(M)} T(M;1-x,0) &= (-1)^{r(M)} \sum_{A \subseteq E} (-x)^{r(M)-r(A)} (-1)^{|A|-r(A)} \\
&= \sum_{A \subseteq E} x^{r(M)-r(A)} (-1)^{|A|} \\
&= \sum_{K \in L} \underbrace{\left(\sum_{\substack{A \subseteq E \\ \bar{A} = K}} (-1)^{|A|} \right)}_{f(K)} x^{r(M)-r(K)}
\end{aligned}$$

so it suffices to check that $f(K) = \mu_L(\hat{0}, K)$. To do this, we use Möbius inversion on L. For $K \in L$, let

$$g(K) \;=\; \sum_{\substack{A \subseteq E \\ \bar{A} \subseteq K}} (-1)^{|A|}.$$

So $g = f \cdot \zeta$ and $f = g \cdot \mu$ in $I(L)$. Then $g(\hat{0}) = 1$, but if $J \neq \hat{0}$ then $g(J) = 0$, because the sum ranges over all subsets of the atoms lying below K, so by Möbius inversion (this time, (1a)) we have

$$f(K) \;=\; \sum_{J \leq K} \mu(J,K) g(J) \;=\; \mu(\hat{0}, K)$$

as desired. ∎