Monday 2/25

The Incidence Algebra

Many enumerative properties of posets P can be expressed in terms of a ring called its **incidence algebra**.

Definition 1. Let P be a locally finite poset and let $\mathrm{Int}(P)$ denote the set of intervals of P. The **incidence algebra** $I(P)$ is the set of functions $f : \mathrm{Int}(P) \to \mathbb{C}$. I'll abbreviate $f([x,y])$ by $f(x,y)$. (For convenience, we set $f(x,y) = 0$ if $x \not\leq y$.) This is a $\mathbb{C}$-vector space with pointwise addition, subtraction and scalar multiplication. It can be made into an associative algebra by the *convolution product*:

$$(f * g)(x,y) = \sum_{z \in [x,y]} f(x,z)g(z,y).$$

Proposition 1. *Convolution is associative.*

Proof.

$$\begin{aligned}
[(f*g)*h](x,y) &= \sum_{z \in [x,y]} (f*g)(x,z) \cdot h(z,y) \\
&= \sum_{z \in [x,y]} \left(\sum_{w \in [x,z]} f(x,w)g(w,z) \right) h(z,y) \\
&= \sum_{w \in [x,y]} f(x,w) \left(\sum_{z \in [w,y]} g(w,z)h(z,y) \right) \\
&= \sum_{w \in [x,y]} f(x,w) \cdot (g*h)(w,y) \\
&= [f*(g*h)](x,y).
\end{aligned}$$

□

Proposition 2. *$f \in I(P)$ is invertible if and only if $f(x,x) \neq 0$ for all x.*

Proof. If f is invertible with inverse g, then $(f*g)(x,x) = f(x,x)g(x,x)$ for all x, so in particular $f(x,x) \neq 0$.

OTOH, if $f(x,x) \neq 0$, then we can define a left inverse g inductively: $g(x,x) = 1/f(x,x)$, and for $x < y$, we want to have

$$\begin{aligned}
(g*f)(x,y) = 0 &= \sum_{x \leq z \leq y} g(x,z)f(z,y) \\
&= g(x,y)f(x,x) + \sum_{x \leq z < y} f(x,z)g(z,y)
\end{aligned}$$

so define

$$g(x,y) = -\frac{1}{f(x,x)} \sum_{x \leq zy} g(x,z)f(z,y).$$

□

The identity element of $I(P)$ is the Kronecker delta function:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The *zeta function* is defined as

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x \not\leq y \end{cases}$$

and the *eta function* is

$$\eta(x,y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } x \not< y, \end{cases}$$

i.e., $\eta = \zeta - \delta$.

Principle: Counting various structures in P corresponds to computation in $I(P)$.

For example, look at powers of ζ:

$$\begin{aligned}
\zeta^2(x,y) &= \sum_{z\in[x,y]} \zeta(x,z)\zeta(z,y) = \sum_{z\in[x,y]} 1 \\
&= \Big|\{z :\ x \leq z \leq y\}\Big| \\
\zeta^3(x,y) &= \sum_{z\in[x,y]} \sum_{w\in[z,y]} \zeta(x,z)\zeta(z,w)\zeta(w,y) = \sum_{x\leq z\leq w\leq y} 1 \\
&= \Big|\{z,w :\ x \leq z \leq w \leq y\}\Big| \\
\zeta^n(x,y) &= \Big|\{x_1,\dots,x_{n-1} :\ x \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq y\}\Big| \\
&= \text{number of } n\text{-multichains between } x \text{ and } y
\end{aligned}$$

Similarly

$$\begin{aligned}
\eta^n(x,y) &= \Big|\{x_1,\dots,x_{n-1} :\ x < x_1 < x_2 < \cdots < x_{n-1} < y\}\Big| \\
&= \text{number of } n\text{-chains between } x \text{ and } y
\end{aligned}$$

- If P is chain-finite then $\eta^n = 0$ for $n \gg 0$.

The Möbius Function

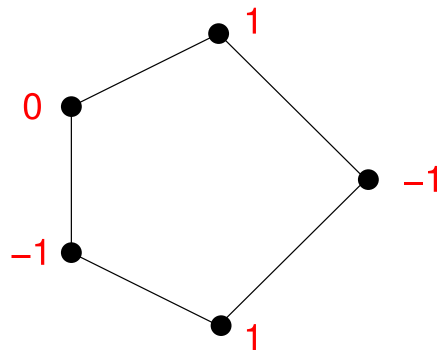
Let P be a poset. We are going to define a function $\mu = \mu_P$ on pairs of comparable elements of P (equivalently, on intervals of P), called the *Möbius function* of P. The definition is recursive:

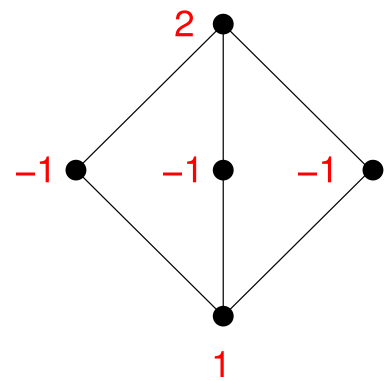
(1) $\mu_P(x,x) = 1$ for all $x \in P$.
(2) If $x \not\leq y$, then $\mu_P(x,y) = 0$.
(3) If $x < y$, then $\mu_P(x,y) = -\sum_{z:\ x\leq z<y} \mu_P(x,z)$.

This is just the construction of Proposition 2 applied to $f = \zeta$. That is, $\mu = \zeta^{-1}$: the Möbius function is the unique function in $I(P)$ satisfying the equations

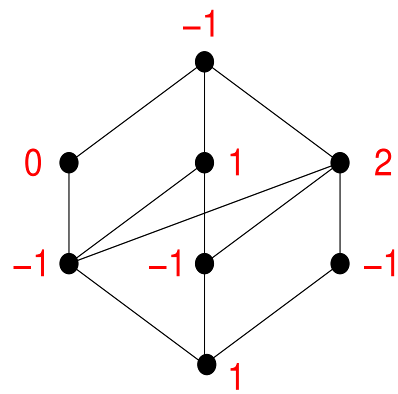
$$\sum_{z:\ x\leq z\leq y} \mu_P(x,z) = \delta(x,y).$$

Example 1. In these diagrams of the posets M_5 and N_5, the red numbers indicate $\mu_P(\hat{0},x)$.





Example 2. In the diagram of the following poset P, the red numbers indicate $\mu_P(\hat{0}, x)$.



Example 3. Let $\mathscr{B}_n$ be the Boolean algebra of rank n and let $A \in \mathscr{B}_n$. I claim that $\mu(\hat{0}, A) = (-1)^{|A|}$.

To see this, induct on $|A|$. The case $|A| = 0$ is clear. For $|A| > 0$, we have

$$\begin{aligned} \mu(\hat{0}, A) = -\sum_{B \subsetneq A} (-1)^{|B|} &= -\sum_{k=0}^{|A|-1} (-1)^k \binom{|A|}{k} \quad \text{(by induction)} \\ &= (-1)^{|A|} - \sum_{k=0}^{|A|} (-1)^k \binom{|A|}{k} \\ &= (-1)^{|A|} - (1-1)^{|A|} \; = \; (-1)^{|A|}. \end{aligned}$$

More generally, if $B \subseteq A$, then $\mu(B, A) = (-1)^{|B \setminus A|}$, because every interval of $\mathscr{B}_n$ is a Boolean algebra.

Even more generally, suppose that P is a product of n chains of lengths $a_1, \dots, a_n$. That is,

$$P = \{x = (x_1, \dots, x_n) \mid 0 \leq x_i \leq a_i \text{ for all } i \in [n]\},$$

ordered by $x \leq y$ iff $x_i \leq y_i$ for all i. Then

$$\mu(\hat{0}, x) = \begin{cases} 0 & \text{if } x_i \geq 2 \text{ for at least one } i; \\ (-1)^s & \text{if } x \text{ consists of } s \text{ 1's and } n - s \text{ 0's.} \end{cases}$$

(The Boolean algebra is the special case that $a_i = 2$ for every i.) This conforms to the definition of Möbius function that you saw in Math 724. This formula is sufficient to calculate $\mu(y, x)$ for all $x, y \in P$, because every interval $[y, \hat{1}] \subset P$ is also a product of chains.

Example 4. We will calculate the Möbius function of the subspace lattice $L = L_n(q)$. Notice that if $X \subset Y \subset \mathbb{F}_q^n$ with $\dim Y - \dim X = m$, then $[X, Y] \cong L_m(q)$. Therefore, it suffices to calculate

$$f(q,n) := \mu(0, \mathbb{F}_q^n).$$

Let $g_q(k,n)$ be the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

Clearly $f(q,1) = \boxed{-1.}$

If $n = 2$, then $g_q(1,2) = \dfrac{q^2-1}{q-1} = q+1$, so $f(q,2) = -1 + (q+1) = \boxed{q.}$

If $n = 3$, then $g_q(1,3) = g_q(2,3) = \dfrac{q^3-1}{q-1} = q^2+q+1$, so

$$\begin{aligned} f(q,3) = \mu(\hat{0},\hat{1}) &= -\sum_{V \subsetneq \mathbb{F}_q^3} \mu(\hat{0}, V) \\ &= -\sum_{k=0}^{2} g_q(k,3) f(q,k) \\ &= -1 - (q^2+q+1)(-1) - (q^2+q+1)(q) \;=\; \boxed{-q^3.} \end{aligned}$$

For $n = 4$:

$$\begin{aligned} f(q,4) &= -\sum_{k=0}^{3} g_q(k,4) f(q,k) \\ &= -1 - \frac{q^4-1}{q-1}(-1) - \frac{(q^4-1)(q^3-1)}{(q^2-1)(q-1)}(q) - \frac{q^4-1}{q-1}(-q^3) \;=\; \boxed{q^6.} \end{aligned}$$

It is starting to look like

$$f(q,n) = (-1)^n q^{\binom{n}{2}}$$

in general, and indeed this is the case. We could prove this by induction now, but there is a slicker proof coming soon.

Why is the Möbius function useful?

- It is the inverse of ζ in the incidence algebra (check this!)
- It generalizes inclusion-exclusion.
- It behaves nicely under poset operations such as product.
- It has geometric and topological applications. Even just the single number $\mu_P(\hat{0},\hat{1})$ tells you a lot about a bounded poset P; it is analogous to the Euler characteristic of a topological space.