Friday 2/8/08

Geometric Lattices and Matroids

Warning: If A is a set and e isn't, then I am going to abuse notation by writing $A \cup e$ and $A \setminus e$ instead of $A \cup \{e\}$ and $A \setminus \{e\}$, when no confusion can arise.

Recall that a **matroid closure operator** on a finite set E is a map $A \mapsto \bar{A}$ on subsets $A \subseteq E$ satisfying

(1a) $\quad A \subseteq \bar{A} = \bar{\bar{A}};$

(1b) $\quad A \subseteq B \implies \bar{A} \subseteq \bar{B};$

(1c) $\quad e \notin \bar{A},\ e \in \overline{A \cup f} \implies f \in \overline{A \cup e}$ (the *exchange condition*).

A **matroid** M is then a set E (the "ground set") together with a matroid closure operator. A closed subset of M (i.e., a set that is its own closure) is called a **flat** of M. The matroid is called **simple** if $\emptyset$ and all singleton sets are closed.

Theorem 1. *1. Let M be a simple matroid with finite ground set E. Let $L(M)$ be the poset of flats of M, ordered by inclusion. Then $L(M)$ is a geometric lattice, under the operations $A \wedge B = A \cap B$, $A \vee B = \overline{A \cup B}$.*

2. Let L be a geometric lattice and let E be its set of atoms. Then the function $\bar{A} = \{e \in E \mid e \leq \bigvee A\}$ is a matroid closure operator on E.

Proof. **For assertion (1)**, we start by showing that $L(M)$ is a lattice. The intersection of flats is a flat (an easy exercise), so the operation $A \wedge B = A \cap B$ makes $L(M)$ into a meet-semilattice. It's bounded (with $\hat{0} = \bar{\emptyset}$ and $\hat{1} = E$), so it's a lattice by [1/25/08, Prop. 2]. Meanwhile, $\overline{A \cup B}$ is the meet of all flats containing both A and B.

By definition of a simple matroid, the singleton subsets of E are atoms in $L(M)$. Every flat is the join of the atoms corresponding to its elements, so $L(M)$ is atomic. The next step is to show that $L(M)$ is semimodular.

Claim: If $F \in L(M)$ and $e \in E \setminus F$, then $F \lessdot F \vee \{e\}$.

Indeed, if $F \subsetneq F' \subseteq F \vee \{e\} = \overline{F \cup \{e\}}$, then for any $f \in F' \setminus F$, we have $e \in F \vee \{f\} \subset F'$ by (1c), so $F' = F \vee \{e\}$, proving the claim.

On the other hand, if $F \lessdot F'$ then $F' = F \vee \{e\}$ for any atom $e \in F' \setminus F$. So we have exactly characterized the covering relations in $L(M)$. It follows that L is ranked, with rank function

$$r(F) = \min \left\{ |B| :\ B \subset E,\ F = \bigvee B \right\}.$$

(Such a set B is called a *basis* of F.)

We now need to show that r satisfies the submodular inequality. Let F, F' be flats and let $G = F \wedge F'$. Let

$$G \lessdot G \vee \{e_1\} \lessdot G \vee \{e_1\} \vee \{e_2\} \lessdot \cdots \lessdot G \vee \{e_1\} \vee \cdots \vee \{e_p\} = F$$
$$G \lessdot G \vee \{e'_1\} \lessdot G \vee \{e'_1\} \vee \{e'_2\} \lessdot \cdots \lessdot G \vee \{e'_1\} \vee \cdots \vee \{e'_q\} = F'$$

be maximal chains, so that

(2) $$r(F) - r(G) = p \quad \text{and} \quad r(F') - r(G) = q.$$

But then $\overline{G \cup \{e_1, \dots, e_p, e'_1, \dots, e'_q\}} = F \vee F'$, so

$$F \ \leq\ F \vee \{e'_1\} \ \leq \cdots \leq\ F \vee \{e'_1\} \vee \cdots \vee \{e'_q\} = F \vee F',$$

where each $\leq$ is either $\lessdot$ or $=$. So $r(F \vee F') - r(G) \leq p + q$, which when combined with (2) implies submodularity.

For assertion (2), it is easy to check that $A \mapsto \bar{A}$ is a closure operator, and that $\bar{A} = A$ for $|A| \leq 1$. So the only nontrivial part is to establish (1c).

Note that if L is semimodular, $e \in L$ is an atom, and $x \not\geq e$, then $x \vee e \gtrdot e$ (because $r(x \vee e) - r(x) \leq r(e) - r(x \wedge e) = 1 - 0 = 1$).

Accordingly, suppose that $e \notin \bar{A}$ but $e \in \overline{A \cup f}$. Let $x = \bigvee A \in L$. Then

$$x \lessdot x \vee f$$

and

$$x < x \vee e \leq x \vee f$$

which implies that $x \vee f = x \vee e$, and in particular $f \leq x \vee e = \overline{A \cup e}$, proving that $A \mapsto \bar{A}$ is a matroid closure operator. □

In view of this bijection, we can describe a matroid on ground set E by the function $A \mapsto r(\bar{A})$, where r is the rank function of the associated geometric lattice. It is standard to abuse notation by calling this function r also. Formally:

Definition 1. A *matroid rank function* on E is a function $r : 2^E \to \mathbb{N}$ satisfying

(3a) $$r(A) \leq |A|; \quad \text{and}$$

(3b) $$r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$$

for all $A, B \subseteq E$.

Example 1. Let $n = |E|$ and $0 \leq k \leq E$, and define

$$r(A) = \min(k, |A|).$$

This clearly satisfies (3a) and (3b). The corresponding matroid is called the *uniform matroid* $U_k(n)$, and has closure operator

$$\bar{A} = \begin{cases} A \text{ if } |A| < k, \\ E \text{ if } |A| \geq k. \end{cases}$$

So the flats of M of the sets of cardinality $< k$, as well as (of course) E itself. Therefore, the lattice of flats looks like a Boolean algebra $\mathscr{B}_n$ that has been truncated at the k^{th} rank. For $n = 3$ and $k = 2$, this lattice is M_5; for $n = 4$ and $k = 3$, it is the following:

If S is a set of n points in general position in $\mathbb{F}^k$, then the corresponding matroid is isomorphic to $U_k(n)$. This sentence is tautological, in the sense that it can be taken as a definition of "general position". Indeed, if $\mathbb{F}$ is infinite and the points are chosen randomly (in some reasonable analytic or measure-theoretic sense), then $L(S)$ will be isomorphic to $U_k(n)$ with probability 1. On the other hand, $\mathbb{F}$ must be sufficiently large (in terms of n) in order for $\mathbb{F}^k$ to have n points in general position.

As for "isomorphic", here's a precise definition.

Definition 2. Let M, M' be matroids on ground sets E, E' respectively. We say that M and M' are **isomorphic**, written $M \cong M'$, if there is a bijection $f : E \to E'$ meeting any (hence all) of the following conditions:

(1) There is a lattice isomorphism $L(M) \cong L(M')$;
(2) $r(A) = r(f(A))$ for all $A \subseteq E$. (Here $f(A) = \{f(a) \mid a \in A\}$.)
(3) $\overline{f(A)} = f(\bar{A})$ for all $A \subseteq E$.

In general, every equivalent definition of "matroid" (and there are several more coming) will induce a corresponding equivalent notion of "isomorphic".

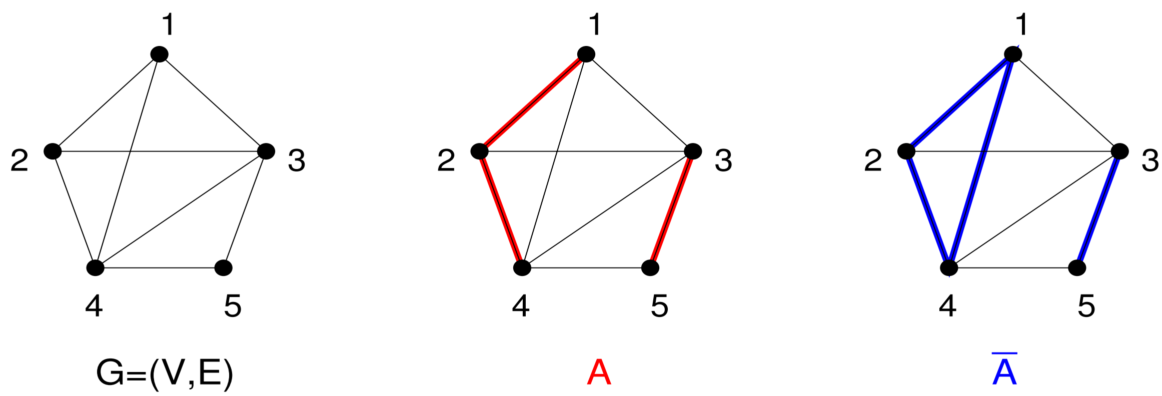
Graphic Matroids

One important application of matroids is in graph theory. Let G be a finite graph with vertices V and edges E. For convenience, we'll write $e = xy$ to mean "e is an edge with endpoints x, y"; this should not be taken to exclude the possibility that e is a loop (i.e., $x = y$) or that some other edge might have the same pair of endpoints.

Definition 3. For each subset $A \subset E$, the corresponding *induced subgraph* of G is the graph $G|_A$ with vertices V and edges A. The *graphic matroid* or *complete connectivity matroid* $M(G)$ on E is defined by the closure operator

$$\bar{A} = \{e = xy \in E \mid x, y \text{ belong to the same component of } G|_A\}. \tag{4}$$

Equivalently, $e = xy \in \bar{A}$ if there is a path between x, e consisting of edges in A (for short, an *A-path*). For example, in the following graph, $14 \in \bar{A}$ because $\{12, 24\} \subset A$.



Proposition 2. *The operator $A \mapsto \bar{A}$ defined by* (4) *is a matroid closure operator.*

Proof. It is easy to check that $A \subseteq \bar{A}$ for all A, and that $A \subseteq B \implies \bar{A} \subseteq \bar{B}$. If $e = xy \in \bar{\bar{A}}$, then x, y can be joined by an $\bar{A}$-path P, and each edge in P can be replaced with an A-path, giving an A-path between x and y.

Finally, suppose $e = xy \notin \bar{A}$ but $e \in \overline{A \cup f}$. Let P be an $(A \cup f)$-path; in particular, $f \in P$. Then $P \cup f$ is a cycle, from which deleting f produces an $(A \cup e)$-path between the endpoints of f. $\square$