# Monday 2/4/08

## Geometric Lattices

**Definition 1.** A lattice is **geometric** if it is (upper) semimodular and atomic.

The term "geometric" comes from the following construction. Let $E$ be a finite set of nonzero vectors in a vector space $V$. Let

$$L(E) = \{W \cap E \mid W \subseteq V \text{ is a vector subspace}\},$$

which is a poset under inclusion. In fact, $L(E)$ is a geometric lattice (homework problem). Its atoms are the singleton sets $\{\{s\} \mid s \in E\}$, and its rank function is $r(Z) = \dim\langle Z\rangle$, where $\langle Z\rangle$ denotes the linear span of the vectors in $Z$.

A closely related construction is the lattice

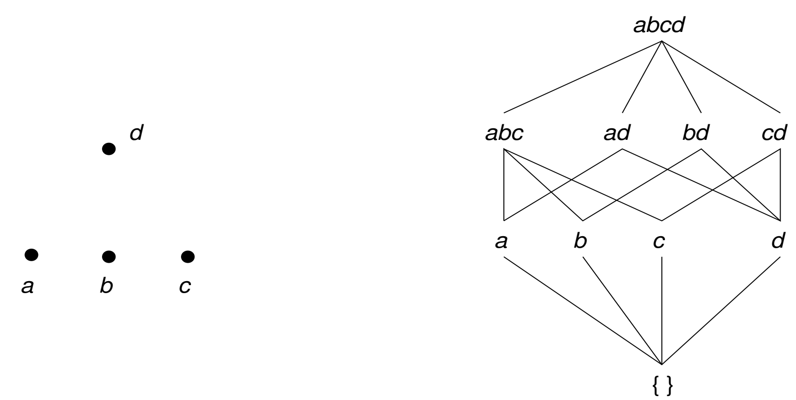
$$L^{\text{aff}}(E) = \{W \cap E \mid W \subseteq V \text{ is an } \underline{\text{affine}} \text{ subspace}\}.$$

(An affine subspace of $V$ is a translate of a vector subspace: for example, a line or plane not necessarily containing the origin.) In fact, any lattice of the form $L^{\text{aff}}(E)$ can be expressed in the form $L(\hat{E})$, where $\hat{E}$ is a certain point set constructed from $E$ (homework problem) However, the rank of $Z \in L^{\text{aff}}(E)$ is one more than the dimension of its affine span, making it more convenient to picture geometric lattices of rank 3.

**Example 1.** The lattices $L_n(q)$ can be obtained as $L(E)$, where $E = \mathbb{F}_q^n \setminus \{\vec{0}\}$. As we know, $L_n(q)$ is actually modular.

**Example 2.** $\Pi_n$ is a geometric lattice. (Homework problem.)

**Example 3.** Let $E$ be the point configuration on the left below. Then $L^{\text{aff}}(E)$ is the lattice on the right (which in this case is modular).




**Example 4.** For arbitrary $E$, the lattice $L(E)$ is not in general modular. For example, let $E = \{w, x, y, z\}$, where $w, x, y, z \in \mathbb{R}^3$ are *in general position*; that is, any three of them form a basis. Then $A = \{w, x\}$ and $B = \{y, z\}$ are flats, and

$$r(A) = r(B) = 2, \quad r(A \wedge B) = 0, \quad r(A \vee B) = 3$$

where $r$ is the rank function on $L(E)$.

Recall that a lattice is *relatively complemented* if, whenever $y \in [x, z] \subseteq L$, there exists $u \in [x, z]$ such that $y \wedge u = x$ and $y \vee u = z$.

**Proposition 1.** *A semimodular lattice $L$ is atomic (equivalently, geometric) if and only if it is relatively complemented.*

Here's the geometric interpretation of being relatively complemented. Suppose that $V$ is a vector space, $L = L(E)$ for some point set $E \subseteq V$, and that $X \subseteq Y \subseteq Z \subseteq E$ are vector subspaces of $V$ spanned by flats of $L(E)$. For starters, consider the case that $X = \{\vec{0}\}$. Then we can choose a basis $B$ of the space $Y$ and extend it to a basis $B'$ of $Z$, and the vector set $B' \setminus B$ spans a subspace of $Z$ that is complementary to $Y$. In the more general case (where we do not require that $X = \{\vec{0}\}$), we can choose a basis $B$ for $X$, extend it to a basis $B'$ of $Y$, and extend $B'$ to a basis $B''$ of $Z$. Then $B \cup (B'' \setminus B')$ spans a subspace $U \subseteq Z$ that is relatively complementary to $Y$, i.e., $U \cap Y = X$ and $U + Y = Z$.

*Sketch of proof of Proposition 1.* Suppose that $L$ is relatively complemented and let $X \in L$. If $X = \hat{0}$ then there is nothing to prove. Otherwise, let $A < X$ be an atom and let $X'$ be a complement for $A$ in $[\hat{0}, X]$. Then $X' < X$ and $X' \vee A$. Replace $X$ with $X'$ and repeat. By induction on rank, we eventually get down to $\hat{0}$, and the list of atoms produced has join $X$.

Now, suppose that $L$ is atomic. Let $y \in [x, z]$, and choose $u \in [x, z]$ such that $y \wedge u = x$ (for instance, $u = x$). If $y \vee u = z$ then we are done. Otherwise, choose an atom $a$ such that $a \leq z$ but $a \not\leq y \vee u$. Set $u' = u \vee a$, so that $u' > u$ (in fact $u' \gtrdot u$). Then, show that $u' \vee y > u \vee y$ and that $u' \wedge y = u \wedge y = x$ (details omitted). By repeatedly replacing $u$ with $u'$ if necessary, we eventually obtain a complement for $y$ in $[x, z]$. $\square$

## Matroids

**Definition 2.** Let $E$ be a finite set. A **closure operator** on $E$ is a map $2^E \to 2^E$ sending $A$ to $\bar{A}$, such that *(i)* $A \subseteq \bar{A} = \bar{\bar{A}}$ and *(ii)* if $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

As a not-quite-trivial consequence

$$\overline{\bar{A} \cap \bar{B}} = \bar{A} \cap \bar{B} \qquad A, B \subseteq E, \tag{1}$$

because $\overline{\bar{A} \cap \bar{B}} \subset \bar{\bar{A}} = \bar{A}$.

A **matroid closure operator** on $E$ is a closure operator satisfying in addition the *exchange axiom*:

$$\text{if } e \notin \bar{A} \text{ but } e \in \overline{A \cup \{f\}}, \text{ then } f \in \overline{A \cup \{e\}}. \tag{2}$$

A **matroid** $M$ is a set $E$ (the "ground set") together with a matroid closure operator. A closed subset of $M$ (i.e., a set that is its own closure) is called a **flat** of $M$. The matroid is called **simple** if the empty set and all singleton sets are closed.

**Example 5.** Let $V$ be a vector space over a field $\mathbb{F}$, and let $E \subseteq V$ be a finite set. Then

$$\bar{A} = \mathbb{F}\text{-span}(A) \cap E$$

is a matroid closure operator on $E$. It is easy to check the conditions for a closure operator. For condition (2), if $e \in \overline{A \cup \{f\}}$, then we have a linear equation

$$e = c_f f + \sum_{a \in A} c_a a, \qquad c_f, c_a \in \mathbb{F}.$$

If $e \notin \bar{A}$, then $c_f \neq 0$, so we can solve for $f$ to express it as a linear combination of the vectors in $A \cup \{e\}$.

The big idea is that the matroid structure records combinatorial information about linear dependence (i.e., which vectors belong to the linear spans of other sets of vectors) without having to worry about the actual coordinates of the vectors.

**Theorem 2.** *1. Let $M$ be a simple matroid with finite ground set $E$. Let $L(M)$ be the poset of closed subsets of $M$, ordered by inclusion. Then $L(M)$ is a geometric lattice.*

*2. Let $L$ be a geometric lattice and let $E$ be its set of atoms. Then the function $\bar{A} = \{e \in E \mid e \leq \bigvee A\}$ is a matroid closure operator on $E$.*