Friday 2/1/08

Modular and Semimodular Lattices

Definition 1. A lattice L is **modular** if for every $x, y, z \in L$ with $x \leq z$,

(1) $$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

It is **(upper) semimodular** if for every $x, y \in L$,

(2) $$x \wedge y \lessdot y \implies x \lessdot x \vee y.$$

Last time, we showed that modular $\implies$ semimodular.

Lemma 1. *Suppose L is semimodular and let $x, y, z \in L$. If $x \lessdot y$, then either $x \vee z = y \vee z$ or $x \vee z \lessdot y \vee z$.*

Proof. Let $w = (x \vee z) \wedge y$. Note that $x \leq w \leq y$. Therefore, either $w = x$ or $w = y$.

- If $w = y$, then $x \vee z \geq y$. So $x \vee z = y \vee (x \vee z) = y \vee z$.
- If $w = x$, then $(x \vee z) \wedge y = x \lessdot y$. Therefore, $(x \vee z) \lessdot (x \vee z) \vee y = y \vee z$. □

Theorem 2. *L is semimodular if and only if it is ranked, with a rank function r satisfying*

(3) $$r(x \vee y) + r(x \wedge y) \leq r(x) + r(y) \qquad \forall x, y \in L.$$

Proof. Suppose that L is a ranked lattice with rank function r satisfying (3). If $x \wedge y \lessdot y$, then $x \vee y > x$ (otherwise $x \geq y$ and $x \wedge y = y$). On the other hand, $r(y) = r(x \wedge y) + 1$, so by (3)

$$r(x \vee y) - r(x) \leq r(y) - r(x \wedge y) = 1$$

which implies that in fact $x \vee y \gtrdot x$.

The hard direction is showing that a semimodular lattice has such a rank function. First, observe that if L is semimodular, then

(4) $$x \wedge y \lessdot x, y \implies x, y \lessdot x \vee y.$$

Denote by $c(L)$ the maximum length* of a chain in L. We will show that L is ranked by induction on $c(L)$.

Base case: If $c(L) = 0$ or $c(L) = 1$, then this is trivial.

Inductive step: Suppose that $c(L) = n \geq 2$. Assume by induction that every semimodular lattice with no chain of length $c(L)$ has a rank function satisfying (3).

First, we show that L is ranked.

Let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{n-1} \lessdot x_n = \hat{1}$ be a chain of maximum length. Let $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{m-1} \lessdot y_m = \hat{1}$ be any maximal† chain in L. We wish to show that $m = n$.

Let $L' = [x_1, \hat{1}]$ and $L'' = [y_1, \hat{1}]$. By induction, these sublattices are both ranked. Moreover, $c(L') = n - 1$.

If $x_1 = y_1$ then we are done by induction, since the interval $L' = [x_1, \hat{1}]$ is a lattice and $c(L') = n - 1$. On the other hand, if $x_1 \neq y_1$, then let $z_1 = x_1 \vee y_1$. By (4), z_1 covers both x_1 and y_1. Let $z_1, z_2, \ldots, \hat{1}$ be a maximal chain in L (thus, in $L' \cap L''$).

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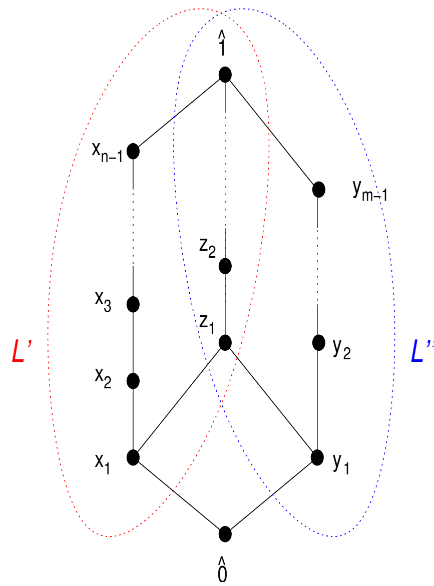
*Remember that the length of a chain is the number of minimal relations in it, which is one less than its cardinality as a subset of L. So, for example, $c(\mathscr{B}_n) = n$, not $n + 1$.

†The terms "maximum" and "maximal" are not synonymous. "Maximum" means "of greatest possible cardinality", while "maximal" means "not contained in any other such object". In general, "maximum" is a stronger condition than "maximal".



Since L' is ranked and $z \gtrdot x_1$, the chain $z_1, \ldots, \hat{1}$ has length $n-2$. So the chain $y_1, z_1, \ldots, \hat{1}$ has length $n-1$.

On the other hand, L'' is ranked and $y_1, y_2, \ldots, \hat{1}$ is a maximal chain, so it also has length $n-1$. Therefore the chain $\hat{0}, y_1, \ldots, \hat{1}$ has length n as desired.

<u>Second</u>, we show that the rank function r of L satisfies (3).

Let $x, y \in L$ and take a maximal chain $x \wedge y = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_{n-1} \lessdot c_n = x$. Note that $n = r(x) - r(x \wedge y)$. Then we have a chain

$$y = c_0 \vee y \leq c_1 \vee y \leq \cdots \leq c_n \vee y = x \vee y.$$

By Lemma 1, each $\leq$ in this chain is either an equality or a covering relation. Therefore, the *distinct* elements $c_i \vee y$ form a maximal chain from y to $x \vee y$, whose length must be $\leq n$. Hence

$$r(x \vee y) - r(y) \leq n = r(x) - r(x \wedge y)$$

and so

$$r(x \vee y) + r(x \wedge y) \leq n = r(x) + r(y).$$

□

The same argument shows that L is lower semimodular if and only if it is ranked, with a rank function satisfying the reverse inequality of (3)

Theorem 3. *L is modular if and only if it is ranked, with a rank function r satisfying*

$$r(x \vee y) + r(x \wedge y) = r(x) + r(y) \qquad \forall x, y \in L. \tag{5}$$

Proof. If L is modular, then it is both upper and lower semimodular, so the conclusion follows by Theorem 2.

On the other hand, suppose that L has rank function r satisfying (5). Let $x \leq z \in L$. We already know that $x \vee (y \wedge z) \leq (x \vee y) \wedge z$. On the other hand,

$$\begin{aligned} r(x \vee (y \wedge z)) &= r(x) + r(y \wedge z) - r(x \wedge y \wedge z) \\ &= r(x) + r(y) + r(z) - r(y \vee z) - r(x \wedge y \wedge z) \\ &\geq r(x) + r(y) + r(z) - r(x \vee y \vee z) - r(x \wedge y) \\ &= r(x \vee y) + r(z) - r(x \vee y \vee z) \qquad = r((x \vee y) \wedge z), \end{aligned}$$

implying (1). □

Geometric Lattices

Recall that a lattice is *atomic* if every element is the join of atoms.

Definition 2. A lattice is **geometric** if it is (upper) semimodular and atomic.

The term "geometric" comes from the following construction. Let E be a finite set of nonzero vectors in a vector space V. Let

$$L(E) = \{W \cap E \mid W \subseteq V \text{ is a vector subspace}\},$$

which is a poset under inclusion. In fact, $L(E)$ is a geometric lattice (homework problem). Its atoms are the singleton sets $\{\{s\} \mid s \in E\}$, and its rank function is $r(Z) = \dim\langle Z\rangle$, where $\langle Z\rangle$ denotes the linear span of the vectors in Z.

A closely related construction is the lattice

$$L^{\text{aff}}(E) = \{W \cap E \mid W \subseteq V \text{ is an \underline{affine} subspace}\}.$$

(An affine subspace of V is a translate of a vector subspace: for example, a line or plane not necessarily containing the origin.) In fact, any lattice of the form $L^{\text{aff}}(E)$ can be expressed in the form $L(\hat{E})$, where $\hat{E}$ is a certain point set constructed from E (homework problem) However, the rank of $Z \in L^{\text{aff}}(E)$ is <u>one more than</u> the dimension of its affine span, making it more convenient to picture geometric lattices of rank 3.

Example 1. Let E be the point configuration on the left below. Then $L^{\text{aff}}(E)$ is the lattice on the right (which in this case is modular).