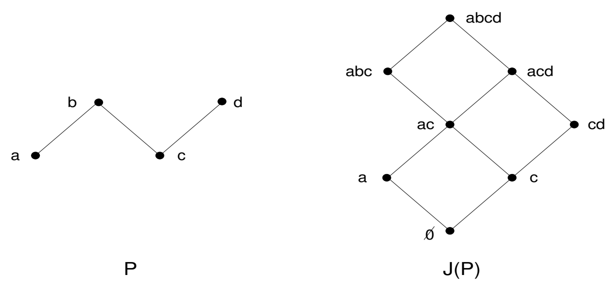
Monday 1/28/08

Birkhoff's Theorem

Definition: A lattice L is **distributive** if the following two equivalent conditions hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in L,$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \in L.$$

Recall that an **(order) ideal** of P is a set $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. The poset $J(P)$ of all order ideals of P (ordered by containment) is a distributive lattice. It is a sublattice of the Boolean algebra $\mathscr{B}_n$ (where $n = |P|$), and is itself ranked, of rank n (i.e., $r(\hat{1}) = n$), because it is possible to build a chain of order ideals by adding one element at a time.



Definition: The ideal **generated** by $x_1, \ldots, x_n$ is

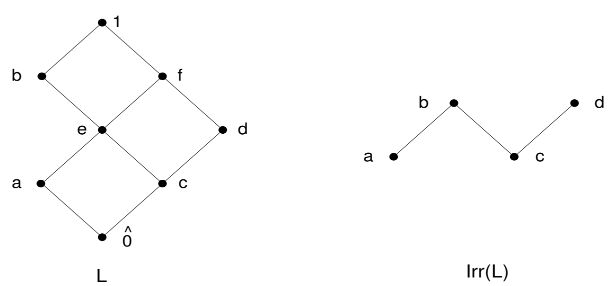
$$\langle x_1, \ldots, x_n \rangle \ := \ \{y \in L \mid y \leq x_i \ \text{ for some } i\}.$$

So, e.g., $\langle a, d \rangle = \{a, c, d\}$ in the lattice above.

Definition: Let L be a lattice. An element $x \in L$ is **join-irreducible** if it cannot be written as the join of two other elements. That is, if $x = y \vee z$ then either $x = y$ or $x = z$. The subposet (not sublattice!) of L consisting of all join-irreducible elements is denoted $\mathrm{Irr}(L)$.

Provided that L has no infinite descending chains, every element of L can be written as the join of join-irreducibles (but not necessarily uniquely; e.g., M_5).

All atoms are join-irreducible, but not all join-irreducible elements need be atoms. An extreme (and slightly trivial) example is a chain: *every* element is join-irreducible, but there is only one atom. As a less trivial example, in the lattice below, a, b, c, d are all join-irreducible, although the only atoms are a and c.



Theorem 1 (**Birkhoff 1933; Fundamental Theorem of Finite Distributive Lattices (FTFDL)**). *Up to isomorphism, the finite distributive lattices are exactly the lattices $J(P)$, where P is a finite poset. Moreover, $L \cong J(\mathrm{Irr}(L))$ for every lattice L and $P \cong \mathrm{Irr}(J(P))$ for every poset P.*

Lemma 2. *Let L be a distributive lattice and let $p \in L$ be join-irreducible. Suppose that $p \leq a_1 \vee \cdots \vee a_n$. Then $p \leq a_i$ for some i.*

Proof. By distributivity we have

$$p = p \wedge (a_1 \vee \cdots \vee a_n) = (p \wedge a_1) \vee \cdots \vee (p \wedge a_n)$$

and since p is join-irreducible, it must equal $p \wedge a_i$ for some i, whence $p \leq a_i$. $\square$

(Analogue: If a prime p divides a product of positive numbers, then it divides at least one of them. This is in fact exactly what Lemma 2 says when applied to the divisor lattice D_n.)

Proposition 3. *Let L be a distributive lattice. Then every $x \in L$ can be written uniquely as an irredundant join of join-irreducible elements.*

Proof. We have observed above that any element in a finite lattice can be written as an irredundant join of join-irreducibles, so we have only to prove uniqueness. So, suppose that we have two irredundant decompositions

$$x = p_1 \vee \cdots \vee p_n = q_1 \vee \cdots \vee q_m \tag{1}$$

with $p_i, q_j \in \mathrm{Irr}(L)$ for all i, j.

By Lemma 1, $p_1 \leq q_j$ for some j. Again by Lemma 1, $q_j \leq p_i$ for some i. If $i \neq 1$, then $p_1 \leq p_i$, which contradicts the fact that the p_i form an antichain. Therefore $p_1 = q_j$. Replacing p_1 with any join-irreducible appearing in (1) and repeating this argument, we find that the two decompositions must be identical. $\square$

Sketch of proof of Birkhoff's Theorem. The lattice isomorphism $L \to J(\mathrm{Irr}(L))$ is given by

$$\phi(x) = \langle p \mid p \in \mathrm{Irr}(L),\ p \leq x \rangle.$$

Meanwhile, the join-irreducible order ideals in P are just the principal order ideals, i.e., those generated by a single element. So the poset isomorphism $P \to \mathrm{Irr}(J(P))$ is given by

$$\psi(y) = \langle y \rangle.$$

These facts need to be checked (as a homework problem).

Corollary 4. *Every distributive lattice is isomorphic to a sublattice of a Boolean algebra (whose atoms are the join-irreducibles in L).*

Corollary 5. *Let L be a finite distributive lattice. TFAE:*

1. *L is a Boolean algebra;*
2. $\mathrm{Irr}(L)$ *is an antichain;*
3. *L is atomic (i.e., every element in L is the join of atoms).*
4. *Every join-irreducible element is an atom;*
5. *L is* complemented. *That is, for each $x \in L$, there exists $y \in L$ such that $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$.*
6. *L is* relatively complemented. *That is, whenever $x \leq y \leq z$ in L, there exists $u \in L$ such that $y \vee u = z$ and $y \wedge u = x$.*

Proof. <u>(6) $\implies$ (5)</u> Trivial.

<u>(5) $\implies$ (4)</u> Suppose that L is complemented, and suppose that $z \in L$ is a join-irreducible that is not an atom. Let x be an atom in $[\hat{0}, z]$, and let y be the complement of x. Then

$$\begin{aligned}(x \vee y) \wedge z &= \hat{1} \wedge z = z \\ &= (x \wedge z) \vee (y \wedge z) = x \vee (y \wedge z),\end{aligned}$$

by distributivity. Since z is join-irreducible, we must have $y \wedge z = z$, i.e., $y \geq z$. But then $y > x$ and $y \wedge x = x \neq \hat{0}$, a contradiction.

<u>(4) $\iff$ (3)</u> Trivial.

<u>$(4) \implies (2)$</u> Atoms are clearly incomparable.

<u>$(2) \implies (1)$</u> By FTFDL, since $L = J(\mathrm{Irr}(L))$.

<u>$(1) \implies (6)$</u> If $X \subseteq Y \subseteq Z$ are sets, then let $U = X \cup (Y \setminus Z)$. Then $Y \cap U = X$ and $Y \cup U = Z$. $\square$

- We could dualize all of this: show that every element in a distributive lattice can be expressed uniquely as the meet of meet-irreducible elements. (This might be a roundabout way to show that distributivity is a self-dual condition.)