Friday 1/25/08

Lattices

Definition: A poset L is a **lattice** if every finite subset of $x, y \in L$ have a unique **meet** $x \wedge y$ and **join** $x \vee y$. That is,

$$x \wedge y = \max\{z \in L \mid z \leq x, y\},$$
$$x \vee y = \min\{z \in L \mid z \geq x, y\}.$$

Note that, e.g., $x \wedge y = x$ if and only if $x \leq y$. These operations are commutative and associative, so for any finite $M \subset L$, the meet $\wedge M$ and join $\vee M$ are well-defined elements of L. In particular, every finite lattice is bounded (with $\hat{0} = \wedge L$ and $\hat{1} = \vee L$).

Proposition 1 (Absorption laws). *Let L be a lattice and $x, y \in L$. Then $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Proof left to the reader.)*

Proposition 2. *Let P be a poset that is a meet-semilattice (i.e., every nonempty $B \subseteq P$ has a well-defined meet $\wedge B$) and has a $\hat{1}$. Then P is a lattice (i.e., every finite nonempty subset of P has a well-defined join).*

Proof. Let $A \subseteq P$, and let $B = \{b \in P \mid b \geq a \text{ for all } a \in A\}$. Note that $B \neq \emptyset$ because $\hat{1} \in B$. I claim that $\wedge B$ is the unique least upper bound for A. First, we have $\wedge B \geq a$ for all $a \in A$ by definition of B and of meet. Second, if $x \geq a$ for all $a \in A$, then $x \in B$ and so $x \geq \wedge B$, proving the claim. $\square$

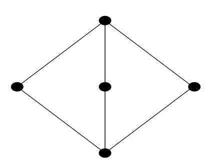
Definition 1. Let L be a lattice. A *sublattice* of L is a subposet $L' \subset L$ that (a) is a lattice and (b) inherits its meet and join operations from L. That is, for all $x, y \in L'$, we have

$$x \wedge_{L'} y = x \wedge_L y \quad \text{and} \quad x \vee_{L'} y = x \vee_L y.$$

Example 1 (The subspace lattice). Let q be a prime power, let $\mathbb{F}_q$ be the field of order q, and let $V = \mathbb{F}_q^n$ (a vector space of dimension n over $\mathbb{F}_q$). The *subspace lattice* $L_V(q) = L_n(q)$ is the set of all vector subspaces of V, ordered by inclusion. (We could replace $\mathbb{F}_q$ with any old field if you don't mind infinite posets.)

The meet and join operations on $L_n(q)$ are given by $W \wedge W' = W \cap W'$ and $W \vee W' = W + W'$. We could construct analogous posets by ordering the (normal) subgroups of a group, or the prime ideals of a ring, or the submodules of a module, by inclusion. (However, these posets are not necessarily ranked, while $L_n(q)$ is ranked, by dimension.)

The simplest example is when $q = 2$ and $n = 2$, so that $V = \{(0,0), (0,1), (1,0), (1,1)\}$. Of course V has one subspace of dimension 2 (itself) and one of dimension 0 (the zero space). Meanwhile, it has three subspaces of dimension 1; each consists of the zero vector and one nonzero vector. Therefore, $L_2(2) \cong M_5$.



Note that $L_n(q)$ is self-dual, under the anti-automorphism $W \to W^\perp$. (An *anti-automorphism* is an isomorphism $P \to P^*$.)

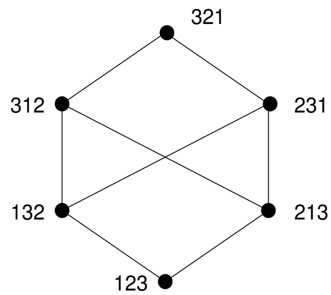
Example 2 (Bruhat order and weak Bruhat order). Let $\mathfrak{S}_n$ be the set of permutations of $[n]$ (i.e., the symmetric group). Write elements of $\mathfrak{S}_n$ as strings $\sigma_1\sigma_2\cdots\sigma_n$ of distinct digits, e.g., $47182635 \in \mathfrak{S}_8$. Impose a partial order on $\mathfrak{S}_n$ defined by the following covering relations:

(1) $\sigma \lessdot \sigma'$ if σ' can be obtained by swapping σ_i with σ_{i+1}, where $\sigma_i < \sigma_{i+1}$. For example,

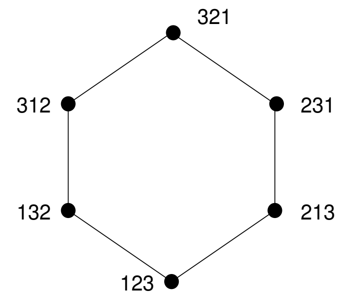
$$4718\underline{26}35 \lessdot 4718\underline{62}35 \quad \text{and} \quad 4\underline{71}82635 \gtrdot 4\underline{17}82635.$$

(2) $\sigma \lessdot \sigma'$ if σ' can be obtained by swapping σ_i with σ_j, where $i < j$ and $\sigma_j = \sigma_i + 1$. For example,
$$4718\underline{2}6\underline{3}5 \lessdot 4718\underline{3}6\underline{2}5.$$

If we only use the first kind of covering relation, we obtain the **weak Bruhat order**.



Bruhat order



Weak Bruhat order

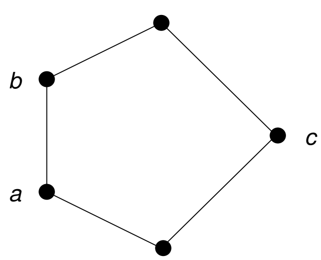
The Bruhat order is not in general a lattice, while the weak order is (although this fact is nontrivial). By the way, we could replace $\mathfrak{S}_n$ with any Coxeter group (although that's a whole 'nother semester).

Both posets are graded and self-dual, and have the same rank function, namely the number of **inversions**:
$$r(\sigma) = \Big|\big\{\{i,j\} \mid i < j \text{ and } \sigma_i > \sigma_j\big\}\Big|.$$
The rank-generating function is a very nice polynomial called the **q-factorial**:
$$F_{\mathfrak{S}_n}(q) = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

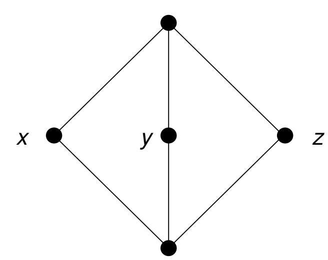
Distributive Lattices

Definition: A lattice L is **distributive** if the following two equivalent conditions hold:
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x,y,z \in L,$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x,y,z \in L.$$
(Proving that these conditions are equivalent is not too hard but is not trivial; it's a homework problem.)

(1) The Boolean algebra $\mathscr{B}_n$ is a distributive lattice, because the set-theoretic operations of union and intersection are distributive over each other.
(2) M_5 and N_5 are not distributive:



N_5



M_5

$$(a \vee c) \wedge b = b$$
$$(a \wedge b) \vee (a \wedge c) = a$$

$$(x \vee y) \wedge z = z$$
$$(x \wedge z) \vee (y \wedge z) = \hat{0}$$

In particular, the partition lattice Π_n is not distributive for $n \geq 3$ (recall that $\Pi_3 \cong M_5$).

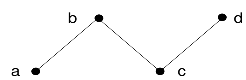
(3) Any sublattice of a distributive lattice is distributive. In particular, Young's lattice Y is distributive because it is locally a sublattice of $\mathscr{B}_n$.

(4) The set D_n of all positive integer divisors of a fixed integer n, ordered by divisibility, is a distributive lattice (proof for homework).

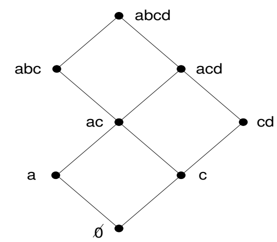
Definition: Let P be a poset. An **(order) ideal** of P is a set $A \subseteq P$ that is closed under going down, i.e., if $x \in A$ and $y \leq x$ then $y \in A$. The poset of all order ideals of P (ordered by containment) is denoted $J(P)$. The order ideal **generated** by $x_1, \ldots, x_n \in P$ is the smallest order ideal containing them, namely

$$\langle x_1, \ldots, x_n \rangle \;:=\; \big\{ y \in P \mid y \leq x_i \text{ for some } i \big\}.$$

By the way, there is a natural bijection between $J(P)$ and the set of antichains of P, since the maximal elements of any order ideal A form an antichain that generates it.



P



J(P)

Proposition: The operations $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ make $J(P)$ into a distributive lattice, partially ordered by set containment.

Sketch of proof: All you have to do is check that $A \cup B$ and $A \cap B$ are in fact order ideals of P. Then $J(P)$ is just a sublattice of the Boolean algebra on P. $\square$