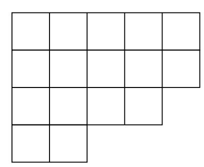
Wednesday 1/23/08

Posets: More Examples

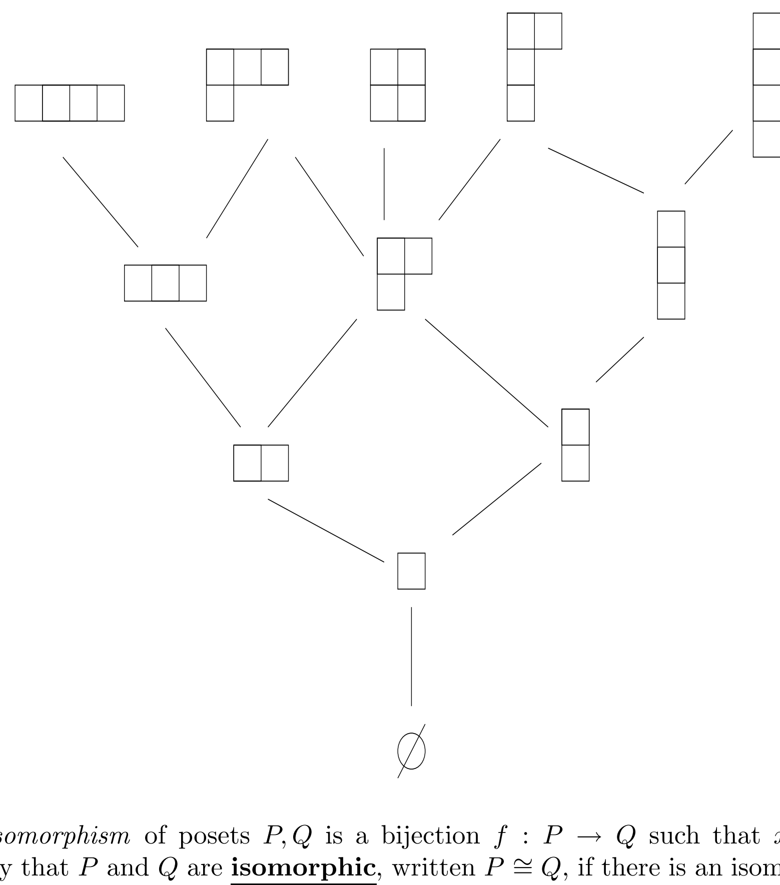
Example 1 (Young's lattice.). A *partition* is a sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of weakly decreasing positive integers: i.e., $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. For convenience, set $\lambda_i = 0$ for all $i > \ell$. Let Y be the set of all partitions, partially ordered by $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i = 1, 2, \dots$.

This is an infinite poset, but it is *locally finite*, i.e., every interval is finite.

There's a nice pictorial way to look at Young's lattice. Instead of thinking about partitions as sequence of numbers, view them as their corresponding **Ferrers diagrams**: northwest-justified piles of boxes whose i^{th} row contains λ_i boxes. For example, 5542 is represented by the following Ferrers diagram:



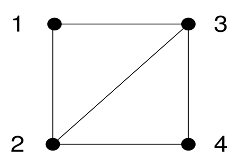
Then $\lambda \geq \mu$ if and only the Ferrers diagram of λ contains that of μ. The top part of the Hasse diagram of Y looks like this:



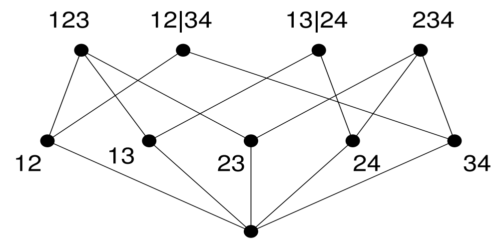
Definition: An *isomorphism* of posets P, Q is a bijection $f : P \to Q$ such that $x \leq y$ if and only if $f(x) \leq f(y)$. We say that P and Q are **isomorphic**, written $P \cong Q$, if there is an isomorphism $P \to Q$. An *automorphism* is an isomorphism from a poset to itself.

Young's lattice Y has a nontrivial automorphism given by *conjugation*. This is most easily described in terms of Ferrers diagrams (reverse the roles of rows and columns). It is easy to check that if $\lambda \geq \mu$, then $\lambda' \geq \mu'$, where the prime denites conjugation.

Example 2 (**The clique poset of a graph**). Let $G = (V, E)$ be a graph with vertex set $[n]$. A *clique* of G is a set of vertices that are pairwise adjacent. Let $K(G)$ be the poset consisting of set partitions all of whose blocks are cliques in G, ordered by refinement.



G



K(G)

This is a *subposet* of Π_n: a subset of Π_n that inherits its order relation This poset is ranked but not graded, since there is not necessarily a $\hat{1}$. Notice that $\Pi_n = K(K_n)$ (the complete graph on n vertices).

Lattices

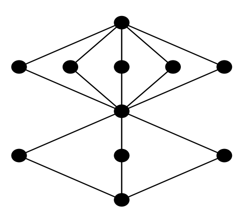
Definition: A poset L is a **<u>lattice</u>** if every $x, y \in L$ have a unique **<u>meet</u>** $x \wedge y$ and **<u>join</u>** $x \vee y$. That is,

$$x \wedge y = \max\{z \in L \mid z \leq x, y\},$$
$$x \vee y = \min\{z \in L \mid z \geq x, y\}.$$

Note that, e.g., $x \wedge y = x$ if and only if $x \leq y$. These operations are commutative and associative, so for any finite $M \subset L$, the meet $\wedge M$ and join $\vee M$ are well-defined elements of L. In particular, every finite lattice is bounded (with $\hat{0} = \wedge L$ and $\hat{1} = \vee L$).

Example 3. The Boolean algebra $\mathscr{B}_n$ is a lattice, with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

Example 4. The *complete graded poset* $P(a_1, \ldots, a_n)$ has $r(\hat{1}) = n + 1$ and $a_i > 0$ elements at rank i for every $i > 0$, with every possible order relation (i.e.,$r(x) > r(y) \implies x > y$).



P(3,1,5)

This is a lattice if and only if no two consecutive a_i's are 2 or greater.

Example 5. The clique poset $K(G)$ of a graph G is in general not a lattice, because join is not well-defined. Meet, however, is well-defined. One could therefore call the clique poset a **<u>meet-semilattice</u>**. It can be made into a lattice by adjoining a brand-new $\hat{1}$ element. In the case that $G = K_n$, the clique poset is a lattice, namely the partition lattice Π_n.

Example 6. Lattices don't have to be ranked. For example, the poset N_5 is a perfectly good lattice.