Friday 1/18/08

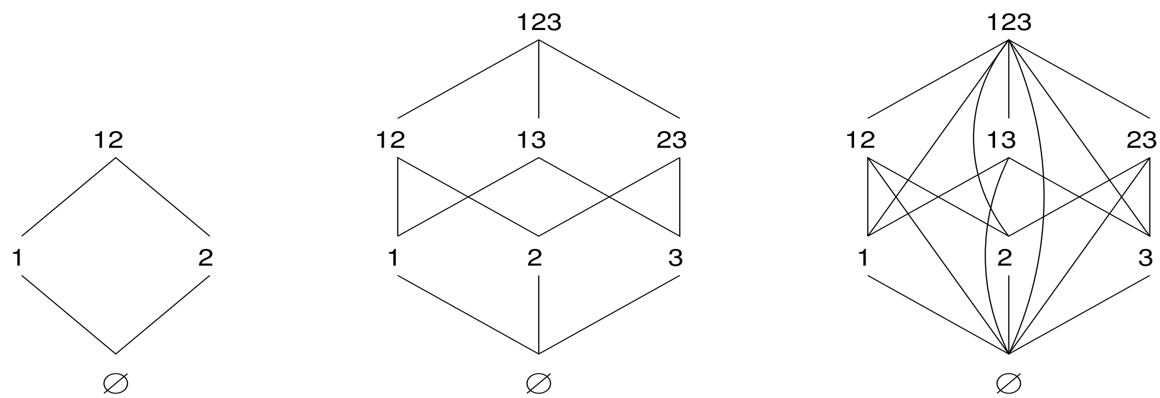
Posets

Definition: A **partially ordered set** or **poset** is a set P equipped with a relation $\leq$ that is reflexive, antisymmetric, and transitive. That is, for all $x, y, z \in P$:

(1) $x \leq x$ (reflexivity).
(2) If $x \leq y$ amd $y \leq x$, then $x = y$ (antisymmetry).
(3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

We'll usually assume that P is finite.

Example 1 (**Boolean algebras**). Let $[n] = \{1, 2, \ldots, n\}$ (a standard piece of notation in combinatorics) and let $\mathscr{B}_n$ be the power set of $[n]$. We can partially order $\mathscr{B}_n$ by writing $S \leq T$ if $S \subseteq T$.



The first two pictures are **Hasse diagrams**. They don't include all relations, just the **covering relations**, which are enough to generate all the relations in the poset. (As you can see on the right, including *all* the relations would make the diagram unnecessarily complicated.)

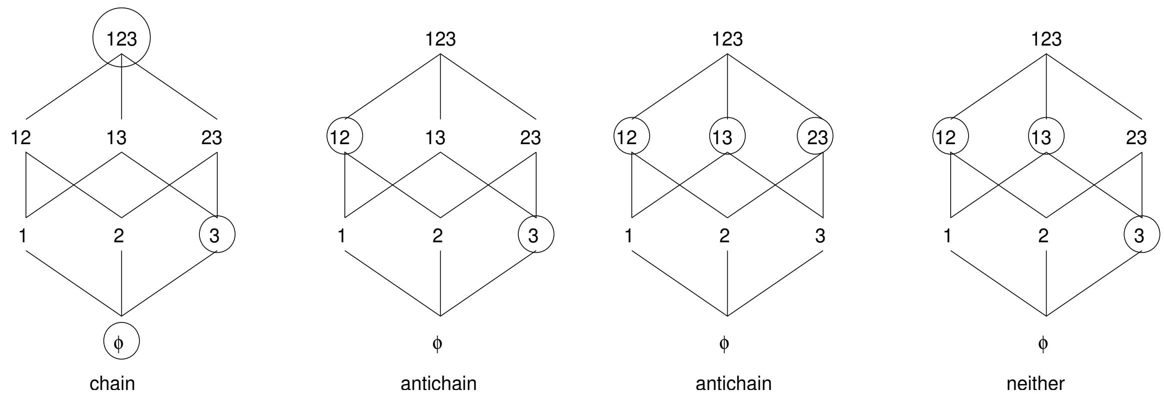
Definitions: Let P be a poset and $x, y \in P$.

- x is **covered** by y, written $x \lessdot y$, if $x < y$ and there exists no z such that $x < z < y$.
- The **interval** from x to y is
$$[x, y] := \{z \in P \mid x \leq z \leq y\}.$$
(This is nonempty if and only if $x \leq y$, and it is a singleton set if and only if $x = y$.)

The Boolean algebra $\mathscr{B}_n$ has a unique minimum element (namely $\emptyset$) and a unique maximum element (namely $[n]$). Not every poset has to have such elements, but if a poset does, we'll call them $\hat{0}$ and $\hat{1}$ respectively.

Definition: A poset that has both a $\hat{0}$ and a $\hat{1}$ is called **bounded**. An element that covers $\hat{0}$ is called an **atom**, and an element that is covered by $\hat{1}$ is called a **coatom**. (For example, the atoms in $\mathscr{B}_n$ are the singleton subsets of $[n]$.)

Definition: A subset $C \subset P$ is called a **chain** if its elements are pairwise comparable. It is called an **antichain** (or *clutter*) if its elements are pairwise incomparable.



Ranked Posets

One of the many nice properties of $\mathscr{B}_n$ is that its elements fall nicely into horizontal slices (sorted by their cardinalities). Whenever $S \lessdot T$, it is the case that $|T| = |S| + 1$. A poset for which we can do this is called a **ranked** poset. However, we can't define "ranked" in this way because it is tautological. Instead:

Definition: A poset is **ranked** if every maximal chain has the same cardinality. A poset is **graded** if it is ranked and bounded. If P is a graded poset, its **rank function** $r : P \to \mathbb{N}$ is defined as

$$r(x) = \text{length of any chain from } \hat{0} \text{ to } x.$$

(Here "length" measures the number of *steps*, not the number of *elements* — i.e., edges rather than vertices in the Hasse diagram.)

Note: "Maximal chain" and "maximum chain" are not synonyms. "Maximal" means "not contained in any other," while "maximum" means "of greatest possible size". Every maximum chain is certainly maximal, but not necessarily vice versa—that's precisely what it means for a poset to be ranked.

An easy consequence of the definition is that if $x \lessdot y$, then $r(y) = r(x) + 1$ (proof left to the reader).

Definition: Let P be a ranked poset with rank function r. The **rank-generating function** of P is the formal power series

$$F_P(q) = \sum_{x \in P} q^{r(x)}.$$

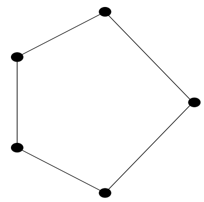
Thus, for each k, the coefficient of q^k is the number of elements at rank k.

We can now say that the Boolean algebra is ranked by cardinality. In particular,

$$F_{\mathscr{B}_n}(q) = \sum_{S \subset [n]} q^{|S|} = (1+q)^n.$$

Of course, if you expand this polynomial out it is palindromic, because the coefficients are a row of Pascal's Triangle. That is, $\mathscr{B}_n$ is **rank-symmetric**. In fact, much more is true. For any poset P, we can define the **dual poset** P^* by reversing all the order relations, or equivalently turning the Hasse diagram upside down. It's not hard to prove that the Boolean algebra is **self-dual**, i.e., $\mathscr{B}_n \cong \mathscr{B}_n^*$, from which it immediately follows that it is rank-symmetric.

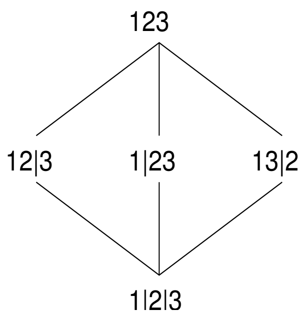
The following is a non-ranked poset (an important example to have around) called N_5.

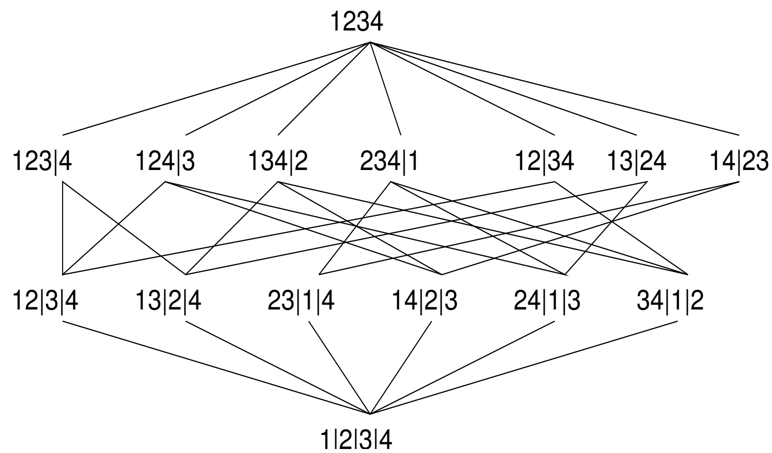


Example 2 (**The partition lattice**). Let Π_n be the poset of all set partitions of $[n]$. E.g., two elements of Π_5 are

$$S = \big\{\{1,3,4\},\ \{2,5\}\big\} \qquad \text{(abbr.: } S = 134|25)$$
$$T = \big\{\{1,3\},\ \{4\},\ \{2,5\}\big\} \qquad \text{(abbr.: } T = 13|4|25)$$

The sets $\{1,3,4\}$ and $\{2,5\}$ are called the *blocks* of S. We can impose a partial order on Π_n by putting $T \leq S$ if every block of T is contained in a block of S; for short, T *refines* S.





- The covering relations are of the form "merge two blocks into one".
- Π_n is graded, with $\hat{0} = 1|2|\cdots|n$ and $\hat{1} = 12\cdots n$. The rank function is $r(S) = n - |S|$.
- The coefficients of the rank-generating function of Π_n are the Stirling numbers of the second kind: $S(n,k)$ = number of partitions of $[n]$ into k blocks. That is,

$$F_n(q) = F_{\Pi_n}(q) = \sum_{k=1}^{n} S(n,k) q^{n-k}.$$

  For example, $F_3(q) = 1 + 3q + q^2$ and $F_4(q) = 1 + 6q + 7q^2 + q^3$.