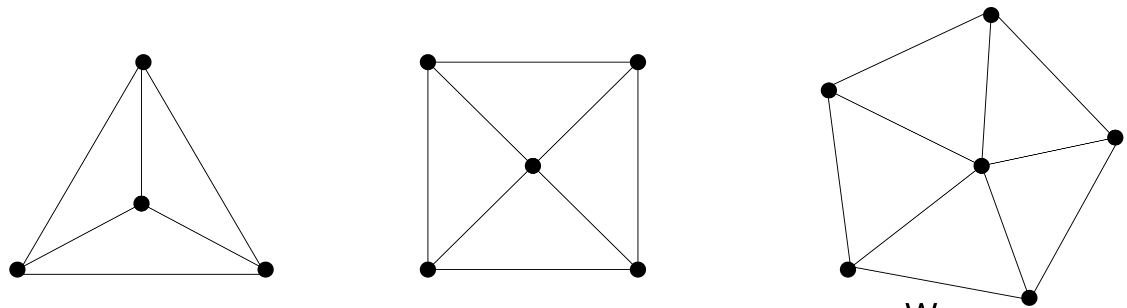
Math 725, Spring 2010
Problem Set #4
Due date: Tuesday, March 9

#1. Let $n \geq 3$. The *n-wheel* W_n is the graph formed by starting with a cycle of length n and introducing a new vertex adjacent to every vertex of the cycle. (Thus W_n has $n+1$ vertices.)



W_3 W_4 W_5

Show that W_n can be decomposed into two spanning trees. Count the number of ways of doing so in terms of n. Find at least one simple graph, not isomorphic to a wheel, with minimum degree ≥ 3, that can be decomposed into two spanning trees. (Graphs of this form turn out to be extremely important in combinatorial rigidity theory — the study of which graphs are "rigid" (like C_3) or "flexible" (like larger cycles).

#2. Construct a simple connected graph G with the following properties: (i) every vertex has degree ≥ 2; (ii) there is a unique vertex v of maximum degree; (iii) no minimum vertex cover contains v. (The point of this example is to show that starting with vertices of high degree is not necessarily the best way to construct a vertex cover.)

#3. [West 3.1.19] Let Y be a finite set and $\mathbf{A} = \{A_1, \ldots, A_m\}$ a family of subsets of Y (not necessarily disjoint). A *system of distinct representatives* (or SDR) for $\mathbf{A}$ is a set of distinct elements $y_1, \ldots, y_m \in Y$ such that $y_i \in A_i$ for all i. Prove that $\mathbf{A}$ has an SDR if and only if $\left|\bigcup_{i \in S} A_i\right| \geq |S|$ for all $S \subseteq [m]$.
(Hint: Transform this into a graph problem.)

#4. [West 3.1.31] Use the König-Egerváry Theorem to prove Hall's Theorem [Theorem 3.1.11 on p.110 of West]. That is, assume that $\alpha'(G) = \beta(G)$ for G bipartite, and use that fact to come up with an alternate proof of Hall's Theorem. You should not assume from the start that Hall's Theorem holds (else this exercise would be trivial!), and your proof should not resemble West's proof of either of these results.

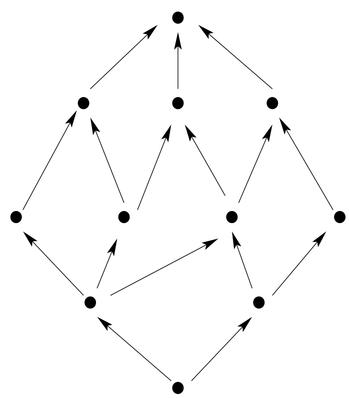
#5. [Schrijver] Let $G = (V, E)$ be a simple graph with $n = n(G)$ and $\delta(G) \geq 2$. Define a *bimatching* to be an edge set $B \subseteq E$ such that no vertex belongs to more than <u>two</u> edges in B, and define a *bicover* to be an edge set $C \subseteq E$ if every vertex belongs to at least two edges in C. Let

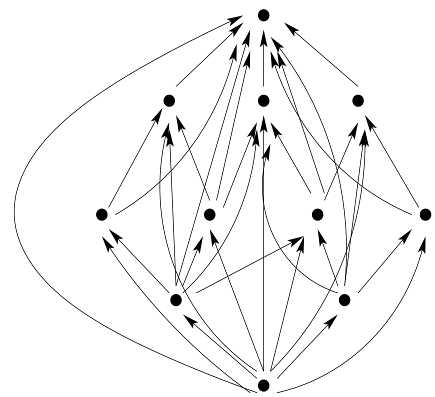
$$\begin{aligned} \tilde{\alpha} &= \tilde{\alpha}(G) = \max\{|B| : B \text{ is a bimatching}\}, \\ \tilde{\beta} &= \tilde{\beta}(G) = \min\{|C| : C \text{ is a bicover}\}. \end{aligned}$$

Prove that $\tilde{\alpha} \leq \tilde{\beta}$ and that $\tilde{\alpha} + \tilde{\beta} = 2n$. (Hint: As in the proof of Gallai's theorem, show separately that $\tilde{\alpha} + \tilde{\beta} \leq 2n$ and that $\tilde{\alpha} + \tilde{\beta} \geq 2n$.)

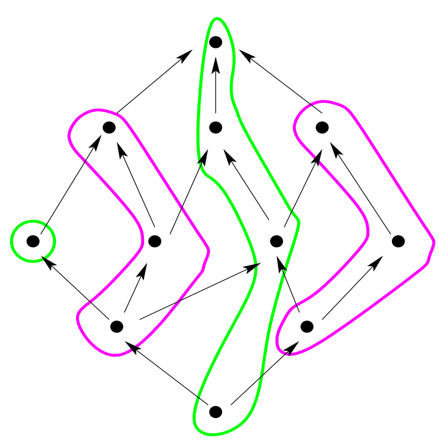
#6. [West 8.4.27(b), modified] A digraph D is called *transitive* if, whenever $u \to v$ and $v \to w$ are edges, then so if $u \to w$.

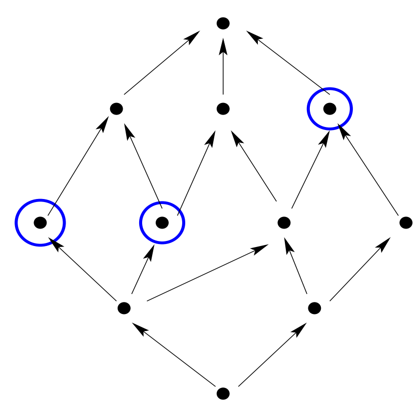
When drawing a transitive digraph, it's enough to specify a set of edges whose transitive closure contains all the other edges. For example, a transitive digraph whose edges include those shown on the left below must in fact contain all the edges shown on the right — but the right-hand picture is disgusting, so it's easier to show the left-hand picture and just remember that it's supposed to be transitive.





A *path cover* in a transitive digraph is a collection of directed paths that partition the vertex set (see figure on left, below — note that the bottom edges of the big green path is not shown, but it is in the transitive closure of the edges shown). An *independent set* is a collection of vertices no two of which are joined by any directed path (see figure on right, below).





Finally, the *split* of a digraph D is the undirected, bipartite graph $G = S(D)$ defined as follows: for each vertex $v \in V(D)$, there are two vertices $v^-, v^+ \in V(G)$, and for each edge $u \to v$ in D, there is an edge u^+v^- in G. (See p. 59 of West.)

(i) Prove that if $\{P_1, \ldots, P_n\}$ is any path cover of a digraph D and $\{v_1, \ldots, v_m\}$ is an independent set in D, then $n \geq m$.

(ii) By applying the König-Egerváry Theorem to the split $S(D)$ of D, prove that the minimum size of a path cover equals the maximum size of an independent set.

This result is known as *Dilworth's Theorem*, and is often stated in the equivalent context of partially ordered sets rather than transitive digraphs. (It's also possible to deduce König-Egerváry from Dilworth.)