Math 725, Spring 2010
Problem Set #3
Due date: Tuesday, February 23

#1. [West 2.3.14] Let $G = (V, E)$ be a connected simple graph with weight function $\mathrm{wt} : E \to \mathbb{R}$. Let C be a cycle in G. Let e be an edge of maximum weight in C; that is, $\mathrm{wt}(e) \geq \mathrm{wt}(e')$ for all $e' \in E(C)$.

(a) Prove that G has some minimum-weight spanning tree not containing e.

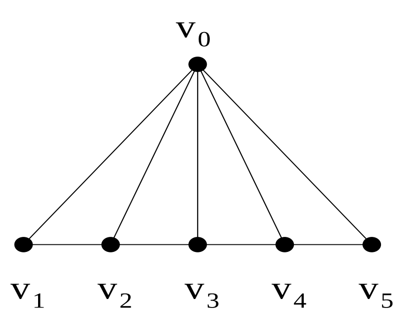
(b) Use part (a) to prove that iteratively deleting a heaviest non-cut-edge until the graph is acyclic produces a minimum-weight spanning tree. (It may help to write out that algorithm more explicitly.)

#2. [West 2.3.18] Explain how to use breadth-first-search to [efficiently] compute the girth of a graph. (Recall that the girth of G is defined as the length of the shortest cycle in G, or ∞ if G is acyclic.)

#3. [West 2.2.16, modified] For each positive integer n, let H_n be the graph defined as follows:

$$\begin{aligned} V(H_n) &= \{v_0,\ v_1,\ \ldots,\ v_n\}, \\ E(H_n) &= \{v_1v_2,\ v_2v_3,\ \ldots,\ v_{n-1}v_n\} \cup \{v_0v_1,\ v_0v_2,\ \ldots,\ v_0v_n\}. \end{aligned}$$

For example, F_5 is shown below.



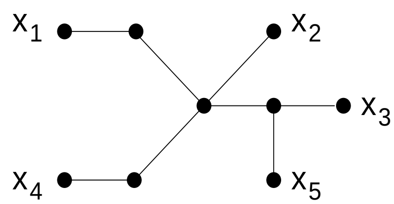
(a) Use the deletion-contraction recurrence (Proposition 2.2.8) to show that $\tau(H_n) = 3\tau(F_{n-1}) - \tau(F_{n-2})$.

(b) Using this recurrence, calculate the numbers $\tau(H_n)$ for $1 \leq n \leq 5$. (They should look familiar.)

(c) Construct a sequence of graphs $G_1, G_2, G_3, \ldots$ with the property that $f_i = \tau(G_i)$ is the i^{th} Fibonacci number; that is, $a_1 = a_2 = 1$ and $a_i = a_{i-1} + a_{i-2}$ for $i \geq 3$. (You should use parts (a) and (b) to do this — for example, it's not fair to define $G_i = C_{f_i}$!)

#4. [West 2.2.18] Use the Matrix-Tree Theorem to compute $\tau(K_{r,s})$ for all numbers r, s. (Hint: Apply elementary row and column operations to the Laplacian matrix of $K_{r,s}$.)

#5. [West 2.1.58, modified] Let T be a tree with m leaves $x_1, \dots, x_m$. The *leaf-distance table* of T is the list of distances between all pairs of leaves. For example, let T be the tree shown below:



Then the leaf-distance table is as follows:

$d(x_1, x_2) = 3$			
$d(x_1, x_3) = 4$	$d(x_2, x_3) = 3$		
$d(x_1, x_4) = 4$	$d(x_2, x_4) = 3$	$d(x_3, x_4) = 4$	
$d(x_1, x_5) = 4$	$d(x_2, x_5) = 3$	$d(x_3, x_5) = 2$	$d(x_4, x_5) = 3$

In this problem, you will prove **Smolenskii's Theorem: If T and S are trees with the same leaf-distance tables, then they are isomorphic.** Specifically: Let S and T be trees with leaves $x_1, \dots, x_m$ and $y_1, \dots, y_m$ respectively, such that $d_S(x_i, x_j) = d_T(y_i, y_j)$ for all i, j. Then there is an isomorphism $f : S \to T$ such that $f(x_i) = y_i$ for every i. (That is, a tree is determined up to isomorphism by its leaf-distance table. In other words, the leaf-distance table is a *complete isomorphism invariant.*)

(a) Suppose that $f : G \to H$ is an isomorphism. Let $u \in V(G)$ and $v = f(u) \in V(H)$. Let G' be the graph constructed by attaching a leaf u' to G at u (that is, $V(G') = V(G) \sqcup \{u'\}$ and $E(G') = E(G) \sqcup \{uu'\}$) and let H' be the graph constructed by attaching a leaf to H at v. Show that $G \cong H$. (This is one of those statements that is intuitively true, but requires careful bookkeeping. In particular, you really have to use the definition of an isomorphism!)

(b) A *stub* of S is a leaf x_i whose unique neighbor in S has degree > 2. Fix a leaf $x_i \in V(S)$ and let w be its stem (unique neighbor). Call x_i a *stub* iff $d_T(w) > 2$. (For instance, in the tree T shown above, the leaves x_2, x_3, x_5 are stubs, but x_1 and x_4 are not.) Show that x_i is a stub if and only if for some j, k,

$$d(x_i, x_j) + d(x_i, x_k) = d(x_j, x_k) + 2.$$

Conclude that x_i is a stub of S if and only if y_i is a stub of T.

(c) Describe the leaves and the leaf-distance table of $S - x_1$ in terms of those of S. (Hint: Consider two cases — either x_i is or is not a stub.)

(d) Prove Smolenskii's theorem by induction on the number of vertices. (Hint: Let x_1' and y_1' be the unique neighbors of x_1 and y_1 respectively. Show that there is an isomorphism $S - x_1 \to T - x_1$ mapping x_1' to y_1', and apply (a). This is easier in the case that the leaves x_1, y_1 are not stubs; if they are stubs, then you need to use (b) to identify x_1' uniquely from the leaf-distance sequence of S.)

Bonus problem: [West 2.3.16] Four people must cross a canyon at night on a fragile bridge. At most two people can be on the bridge at once, and there is only one flashlight (which can cross only by being carried). Kovalevskaia can cross the bridge in 10 minutes, Legendre in 5 minutes, Macaulay in 2 minutes, and Noether in 1 minute. If two people cross together, they move at the speed of the slower person. Oh, by the way, in 18 minutes a flash flood is going to roar down the canyon and wash away the bridge (together with anyone who isn't yet safe on the other side). Can the four people get across in time?